Parasitic SNAIL TWPA Model Derivation

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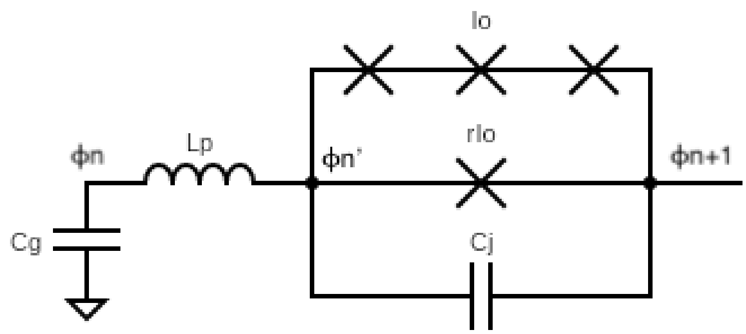

Figure 1: SNAIL TWPA circuit with parasitic inductance.

1 Parasitic Model

1.1 Lagrangian

Let ϕ_s and ϕ_L denote the flux across the SNAIL and parasitic inductor respectively,

$$\phi_s = \varphi_{n+1} - \varphi_{n'}$$

and

$$\phi_L = \varphi_{n'} - \varphi_n$$

where the flux at node n is φ_n such that

$$\partial_t \varphi_n = V_n.$$

For the parasitic circuit we have

$$L = \frac{C_g}{2}\dot{\varphi}_n^2 + \frac{C_J}{2}\dot{\phi}_s^2 - \frac{1}{2L_p}\phi_L^2 - U_{SNAIL}|_{\phi_s} = a\mathcal{L} \tag{1}$$

where L_p is the parasitic inductance. Using Kirschoff's current law to express the current through the inductor in terms of phase across the SNAIL,

$$I_L = I_s|_{\phi_s} + C_J\ddot{\phi}_s = \frac{1}{L_p}\phi_L \tag{2}$$

giving

$$\phi_L = L_p[I_s|_{\phi_s} + C_J\ddot{\phi}_s]. \tag{3}$$

Rewriting the Lagrangian,

$$L = \frac{C_g}{2}\dot{\varphi_n}^2 + \frac{C_J}{2}\dot{\phi_s}^2 - \frac{L_p}{2}[I_s|_{\phi_s} + C_J\ddot{\phi}_s]^2 - U_{SNAIL}|_{\phi_s} = a\mathcal{L} \tag{4}$$

In the limit that the width of each cell is much smaller than the wavelength, the Lagrange density becomes

$$\mathcal{L} \to \frac{C_g}{2}\dot{\Phi}^2 + \frac{C_J}{2}\dot{\Phi}'^2 - \frac{L_p}{2}[I_s|_{\Phi'} + C_J\ddot{\Phi}']^2 - U_{SNAIL}|_{\Phi'}. \tag{5}$$

Next, consider the field Euler Lagrange equation

$$\partial_t\partial_{\dot{\Phi}}\mathcal{L} + \partial_x\partial_{\Phi'}\mathcal{L} = \partial_{\Phi}\mathcal{L}. \tag{6}$$

We have

$$\partial_t\partial_{\dot{\Phi}}\mathcal{L} = C_g\ddot{\Phi}, \tag{7}$$

$$\partial_{\Phi}\mathcal{L} = C_J\ddot{\Phi}'', \tag{8}$$

and

$$\partial_x\partial_{\Phi'}\mathcal{L} = -\partial_x\left[\frac{L_p}{2}\partial_{\Phi'}[I_s(\Phi') + C_J\ddot{\Phi}']^2 + \partial_{\Phi'}U_{SNAIL}(\Phi')\right]$$

$$= \text{-}\partial_x\left[L_p[I_s(\Phi') + C_J\ddot{\Phi}']\partial_{\Phi'}I_s(\Phi') + I_s(\Phi')\right]$$

$$= \text{-}\partial_x\left[I_s(\Phi')(1 + L_p\partial_{\Phi'}I_s(\Phi')) + L_pC_J\ddot{\Phi}'\partial_{\Phi'}I_s(\Phi')\right]. \tag{9}$$

Expanding the current to third order about a minimum ϕ^* and perturbing by ϕ gives

$$\frac{I_s(\phi^* + \phi)}{I_0} \approx \alpha\phi - \beta\phi^2 - \gamma\phi^3 \tag{10}$$

for constant coefficients

$$\alpha = r\cos(\phi^*) + \frac{1}{3}\cos(\phi^* - \phi_{ex}/3), \tag{11}$$

$$\beta = \frac{1}{2}[r\sin(\phi^*) + \frac{1}{9}\sin(\phi^* - \phi_{ex}/3)], \tag{12}$$

$$\gamma = \frac{1}{6}[r\cos(\phi^*) + \frac{1}{27}\cos(\phi^* - \phi_{ex}/3)]. \tag{13}$$

The inductance per cell may be approximated by keeping up to first order terms from the expanded current-phase equation,

$$L \propto \frac{\Phi_0}{2\pi I_0 \alpha}. \tag{14}$$

Then, setting 3WM and 4WM coefficients to

$$g_3 \propto \frac{\beta}{\alpha} \tag{15}$$

and

$$g_4 \propto \frac{\gamma}{\alpha} \tag{16}$$

Substituting into the equation of motion:

$$\begin{aligned}\partial_x \partial_{\Phi'} \mathcal{L} = -\frac{1}{L}\partial_x[(1 + \frac{L_p}{L} + 2L_p C_J \ddot{\Phi}' g_3)\Phi' + (g_3 + 3\frac{L_p}{L} g_3(1+g_4) + 3L_p C_J g_4)\Phi'^2 \\ +(g_4 + \frac{L_p}{L}(2g_3^2 + g_4))\Phi'^3 + 2\frac{L_p}{L} g_3 g_4 \Phi'^4 + \frac{L_p}{L} g_4^2 \Phi'^5]\end{aligned}$$

1.2 Ideal Four Wave Mixing

We assume the system contains a degenerate four-wave mixing (DFWM), process in which two pump photons excite the system which then, via stimulated emission, releases signal and idler tones. Just consider 4WM $g_3 = 0$

$$\partial_x \partial_{\Phi'} \mathcal{L} = -\frac{1}{L}\partial_x[(1 + \frac{L_p}{L})\Phi' + (3L_p C_J g_4)\Phi'^2 + g_4(1 + \frac{L_p}{L})\Phi'^3 + \frac{L_p}{L} g_4^2 \Phi'^5] \tag{17}$$

which can be rewritten as

$$\partial_x \partial_{\Phi'} \mathcal{L} = -\frac{1 + L_p/L}{L}\Phi'' - \mathcal{N}. \tag{18}$$

where $\mathcal{N}$ is the non-linear component of the equation of motion

$$\mathcal{N} = \frac{1}{L}\partial_x[(3L_p C_J g_4)\Phi'^2 + g_4(1 + \frac{L_p}{L})\Phi'^3 + \frac{L_p}{L} g_4^2 \Phi'^5] \tag{19}$$

Parasitic Effect 1: A reduced effective inductance which changes the characteristic frequency of the transmission line and plasma frequency of the junctions. Consequences include inaccurate design parameters and impedance mismatch.

$$L_{\text{eff}} = \frac{L}{1 + L_p/L} < L \tag{20}$$

The propagation equation is therefore

$$C_g \ddot{\Phi} - C_J \ddot{\Phi}'' - \frac{1}{L_{\text{eff}}}\Phi'' = \mathcal{N}. \tag{21}$$

The ansatz, ignoring all 3-wave mixing effects, is a forward propagating wave of the form

$$\phi = \frac{1}{2}\sum_{m \in p,s,i}[A_m(x)e^{i(k_m x + \omega_m t)} + c.c]. \tag{22}$$

1.2.1 Linear Parasitic Effects

Substituting Eq. 22 into the left side of Eq. 21,

$$\frac{1}{2}\left[-C_g\omega_m^2\phi_m + C_J\omega_m^2\partial_x^2(A_m(x)e^{i(k_mx+\omega_mt)}) - \frac{1}{L_{\text{eff}}}\partial_x^2(A_m(x)e^{i(k_mx+\omega_mt)})\right] \tag{23}$$

or

$$\frac{1}{2}\left[-C_g\omega_m^2A_m + (C_J\omega_m^2 - \frac{1}{L_{\text{eff}}})(A_m'' + 2ik_mA_m' - k_m^2A_m)\right]e^{i(k_mx+\omega_mt)} \tag{24}$$

Next, we make the Slow Varying Envelope Approximation (SVEA) $|A_m''| \ll |k_mA_m'|$ and express the left side of the Eq. 21 in terms of the ansatz amplitudes:

$$\frac{1}{2}\left[(-C_g\omega_m^2 - C_J\omega_m^2k_m^2 + \frac{k_m^2}{L_{\text{eff}}})A_m + 2ik_m(C_J\omega_m^2 - \frac{1}{L_{\text{eff}}})A_m'\right]e^{i(k_mx+\omega_mt)} \tag{25}$$

In the absence of nonlinear effect ($g_3 = g_4 = 0$ or $\mathcal{N} = 0$), Eq. 25 equals 0. Defining the wave vector in terms of a linear dispersion relation requires the spatial derivative of the amplitude to vanish, therefore to satisfy the absence of nonlinear effects, term coefficient of amplitude must also equal 0, giving

$$\frac{k_m^2}{L_{\text{eff}}} - C_g\omega_m^2 - C_Jk_m^2\omega_m^2 = 0, \tag{26}$$

or

$$k_m = \frac{\omega_m\sqrt{L_{\text{eff}}C_g}}{\sqrt{1 - L_{\text{eff}}C_J\omega_m^2}} = \frac{\omega_m}{\tilde{\omega}_0\sqrt{1 - \omega_m^2/\tilde{\omega}_J^2}}. \tag{27}$$

where tilde represents the corrected frequencies due to the parasitic inductance. Eq. 25 reduces to

$$-\frac{iC_g\omega_m^2}{k_m}e^{i(k_mx+\omega_mt)}A_m' \tag{28}$$

1.2.2 Nonlinear Parasitic Effects

Now consider the nonlinear component (right side) of Eq. 21:

$$\mathcal{N} = \frac{1}{L}\partial_x(\mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N}_5). \tag{29}$$

where

$$\mathcal{N}_2 = 3L_pC_Jg_4\Phi'^2,$$
$$\mathcal{N}_3 = g_4(1 + \frac{L_p}{L})\Phi'^3,$$
$$\mathcal{N}_5 = \frac{L_p}{L}g_4^2\Phi'^5.$$

Parasitic Effect 2: There exists an unwanted second order non-linearity proportional to L_p/L, corresponding to 3WM and second harmonic generation. Consequences of upconversion include reduced pump efficiency and loss of signal tones from the amplification band.

We substitute the ansatz and make the following assumptions:

1. Under SVEA, the amplitudes are slowly varying compared to their magnitude: $|A'_m| \ll |k_m A_m|$
2. The pump amplitude is much greater than either the signal or the idler: $|A_{s,i}| \ll |A_p|$.
3. We ignore $\mathcal{N}_5$ because $L_p/L^2 \ll 1$.

$$\frac{1}{L}\partial_x \mathcal{N}_2(\phi) = 2g_{3,p}\Phi'\Phi''$$
$$= \frac{g_{3,p}}{2}\sum_{m,n\in\{p,s,i\}}(ik_m A_m e^{i(k_m x+\omega_m t)} + c.c)(-k_n^2 A_n e^{i(k_n x+\omega_n t)} + c.c)$$

$$\frac{1}{L}\partial_x \mathcal{N}_3(\phi) = 3g_{4,p}\Phi'^2\Phi''$$
$$= \frac{3g_{4,p}}{8}\sum_{m,n,l\in\{p,s,i\}}(ik_m A_m e^{i(k_m x+\omega_m t)} + c.c)(ik_n A_n e^{i(k_n x+\omega_n t)} + c.c)(-k_l^2 A_l e^{i(k_l x+\omega_l t)} + c.c)$$

where $g_{3,p} = \frac{3L_p C_J g_4}{L}$ and $g_{4,p} = \frac{g_4}{L}(1 + \frac{L_p}{L})$ are the parasitic 3WM and 4WM terms, respectively. When concerned with the pump propagation, our second assumption allows us to ignore terms proportional to the amplitude of the signal and idler tones. Considering only $m = n = p$ and setting Eq. 28 to the remaining nonlinear terms, the pump propagation equation becomes

$$\frac{-iC_g\omega_p^2}{k_p}e^{i(k_p x+\omega_p t)}A'_p + c.c = \frac{-ig_{3,p}}{2}k_p^3(A_p^2 e^{2i(k_p x+\omega_p t)} + c.c) + \frac{3g_{4,p}}{8}k_p^4(A_p^3 e^{3i(k_p x+\omega_p t)} + A_p^2 A_p^* e^{i(k_p x+\omega_p t)} + c.c) \tag{30}$$

$$A'_p e^{i(k_p x+\omega_p t)} - \frac{g_{3,p}}{2C_g\omega_p^2}k_p^4 A_p^2 e^{2i(k_p x+\omega_p t)} - \frac{3ig_{4,p}}{8C_g\omega_p^2}k_p^5(A_p^3 e^{3i(k_p x+\omega_p t)} + A_p^2 A_p^* e^{i(k_p x+\omega_p t)}) + c.c = 0 \tag{31}$$

Here we assume no loss, and ignore both second and third harmonic generation to preserve the initial ansatz. In certain cases neglecting parasitic 3WM is justified by $g_{3,p} \ll g_{4,p}$. For example if $L_p \ll L$, or $L_p < L$ *and* $C_J < C_g$, or $C_J \ll C_g$, the term proportional to $e^{i2\omega_p t}$ becomes a negligible correction. Under the undepleted pump approximation, we assume that the third harmonic is always much weaker than the pump, that is $A_p^3 \ll A_p$ because the pump amplitude is already weak. Multiplying by $e^{-i\omega_p t}$ and integrating over time causes traveling waves to average zero contribution, giving

$$A'_p - \frac{3ig_4 k_p^5}{8L_{\text{eff}}C_g\omega_p^2}A_p|A_p|^2 = 0. \tag{32}$$

Therefore

$$A_p(x) = A_{p,0}e^{i\alpha_p x} \tag{33}$$

where

$$\alpha_p = \frac{3g_4 k_p^5|A_p|^2}{8L_{\text{eff}}C_g\omega_p^2}. \tag{34}$$

The propagation equations for the signal and idler tones, neglecting terms quadratic in signal and idler amplitudes, is given by

$$\sum_{m\in\{s,i\}} \frac{-iC_g\omega_m^2}{k_m} A_m' e^{i(k_m x+\omega_m t)} + c.c =$$

$$\sum_{m\in\{s,i\}} \frac{g_{3,p}}{2}\left(-ik_pk_m(k_p+k_m)A_m e^{i(k_m x+\omega_m t)}\left[A_p e^{i(k_p x+\omega_p t)} + \frac{(k_p-k_m)}{(k_p+k_m)}A_p^* e^{-i(k_p x+\omega_p t)}\right] + c.c\right)$$

$$+\frac{3g_{4,p}}{8}\left(2k_mk_p^3(A_m e^{i(k_m x+\omega_m t)}[A_p^2 e^{i(2k_p x+2\omega_p t)} - c.c] + c.c)\right)$$

$$+\frac{3g_{4,p}}{8}\left(k_m^2k_p^2A_m e^{i(k_m x+\omega_m t)}[A_p^2 e^{i(2k_p x+2\omega_p t)} + |A_p|^2 + c.c]\right) + c.c$$

which can be rewritten as

$$\sum_{m\in\{s,i\}} \frac{\omega_m^2}{k_m} A_m' e^{i(k_m x+\omega_m t)} + c.c =$$

$$\sum_{m\in\{s,i\}} \frac{g_{3,p}k_pk_mk_{m+p}}{2C_g}A_m\left(A_p e^{i(k_{m+p}x+\omega_{m+p}t)} + \frac{k_{p-m}}{k_{p+m}}A_p^* e^{i(k_{m-p}x+\omega_{m-p}t)}\right)$$

$$+\frac{i3g_{4,p}k_mk_p^3}{4C_g}A_m\left(A_p^2 e^{i(k_{m+2p}x+\omega_{m+2p}t)} - A_p^{*2}e^{i(k_{m-2p}x+\omega_{m-2p}t)}\right)$$

$$+\frac{i3g_{4,p}k_m^2k_p^2}{8C_g}A_m\left(A_p^2 e^{i(k_{m+2p}x+\omega_{m+2p}t)} + A_p^{*2}e^{i(k_{m-2p}x+\omega_{m-2p}t)} + 2|A_p|^2 e^{i(k_m x+\omega_m t)}\right) + c.c$$

From conservation of energy, $2\omega_p = \omega_s + \omega_i$, we have

$$\sum_{m\in\{s,i\}} \frac{\omega_m^2}{k_m} A_m' e^{ik_m x}e^{i\omega_m t} + c.c = \sum_{m\in\{s,i\}} \frac{g_{3,p}k_pk_mk_{m+p}}{2C_g}A_m\left(A_p e^{i(k_{m+p}x+\omega_{m+p})t} + A_p^* e^{i(k_{m-p}x+\omega_{m-p}t)}\right)$$

$$+\frac{i3g_{4,p}k_mk_p^3}{4C_g}A_m\left(A_p^2 e^{i(k_{m+2p}x+\omega_{m+s+i}t)} - A_p^{*2}e^{i(k_{m-2p}x+\omega_{m-s-i}t)}\right)$$

$$+\frac{i3g_{4,p}k_m^2k_p^2}{8C_g}A_m\left(A_p^2 e^{i(k_{m+2p}x+\omega_{m+s+i}t)} + A_p^{*2}e^{i(k_{m-2p}x+\omega_{m-s-i}t)} + 2|A_p|^2 e^{i(k_m x+\omega_m t)}\right) + c.c$$

To find the propagation equation for the signal, we multiply by $e^{-i\omega_s t}$ and integrate both sides over time. Only terms that contain $e^{+i\omega_s t}$ remain:

$$\frac{\omega_s^2}{k_s}A_s' e^{ik_s x} = \frac{i3g_{4,p}k_ik_p^3}{4C_g}A_i^*A_p^2 e^{i(k_{2p}-k_i)} - \frac{i3g_{4,p}k_i^2k_p^2}{8C_g}A_i^*A_p^2 e^{i(k_{2p}-k_i)x} + \frac{i3g_{4,p}k_s^2k_p^2}{4C_g}A_s|A_p|^2 e^{ik_s x} \quad (35)$$

Therefore

$$A_s' - \frac{i3g_{4,p}k_s^3k_p^2}{4C_g\omega_s^2}A_s|A_p|^2 - \frac{i3g_{4,p}k_ik_sk_p^2(2k_p-k_i)}{8C_g\omega_s^2}A_i^*A_p^2 e^{i\Delta k_L x} = 0 \quad (36)$$

and similarly for the idler

$$A_i' - \frac{i3g_{4,p}k_i^3k_p^2}{4C_g\omega_i^2}A_i|A_p|^2 - \frac{i3g_{4,p}k_ik_sk_p^2(2k_p-k_s)}{8C_g\omega_i^2}A_s^*A_p^2 e^{i\Delta k_L x} = 0 \quad (37)$$

where we have defined the linear phase mismatch as $\Delta k_L = 2k_p - k_s - k_i$.

Substituting Eq. 33 into Eqs. 36 and 37, we find

$$A_s' - i\alpha_s A_s - i\kappa_s A_i^* e^{i(\Delta k_L + 2\alpha_p)x} = 0 \tag{38}$$

$$A_i' - i\alpha_i A_i - i\kappa_i A_s^* e^{i(\Delta k_L + 2\alpha_p)x} = 0 \tag{39}$$

where we have defined

$$\alpha_s = \frac{2k_s^3 \kappa |A_p|^2}{C_g L_{\text{eff}} \omega_s^2} \tag{40}$$

$$\alpha_i = \frac{2k_i^3 \kappa |A_p|^2}{C_g L_{\text{eff}} \omega_i^2} \tag{41}$$

and

$$\kappa_s = \frac{k_s k_i (2k_p - k_i)\kappa A_{p,0}^2}{L_{\text{eff}} C_g \omega_s^2} \tag{42}$$

$$\kappa_i = \frac{k_s k_i (2k_p - k_s)\kappa A_{p,0}^2}{L_{\text{eff}} C_g \omega_i^2} \tag{43}$$

for

$$\kappa = \frac{3g_4 k_p^2}{8} \tag{44}$$

If we assume the signal and idler tones amplitudes go as

$$A_m = a_m e^{i\alpha_m x} \tag{45}$$

then we can solve the signal propagation equation via substitution:

$$a_s' e^{i\alpha_s x} + i\alpha_s a_s e^{i\alpha_s x} - i\alpha_s a_s e^{i\alpha_s x} - i\kappa_s a_i^* e^{-i\alpha_i x} e^{i(\Delta k_L + 2\alpha_p)x} = 0$$

$$a_s' e^{i\alpha_s x} - i\kappa_s a_i^* e^{i(\Delta k_L + 2\alpha_p - \alpha_i)x} = 0$$

$$a_s' - i\kappa_s a_i^* e^{i\Delta k x} = 0. \tag{46}$$

Similarly for the idler tone,

$$a_i' - i\kappa_i a_s^* e^{i(\Delta k)x} = 0. \tag{47}$$

where we have defined the total phase mismatch as

$$\Delta k = \Delta k_L + \Delta k_{\text{Kerr}} \tag{48}$$

for the Kerr mismatch defined as

$$\Delta k_{\text{Kerr}} = 2\alpha_p - \alpha_s - \alpha_i. \tag{49}$$

The dynamics of the system under the given set of assumptions, including SVEA, undepleted pump, and small signal amplitude, behaves identical to the original kerr phase matched system. If we ignore both ideal and parasitic 3WM ($g_3 = g_{3,p} \approx 0$), we can simulate the device behavior using the original model with the effective inductance defined by Eq. 20. Now we should consider operating in the regime of combined 3- and 4WM as this is required leverage second harmonic generation of reflected signals and suppress measurement back-action.

1.3 Combined Three and Four Wave Mixing

Now consider a system with non-zero g_3 induced by an applied magnetic flux detuned from the $0.5\Phi_0$ operating point. This could be the result of perturbations in the applied field, or intentional detuning for reflection isolation via second harmonic generation. As before, the nonlinear effects come from the spatial variation of the Lagrangian:

$$\partial_x \partial_{\Phi'} \mathcal{L} = -\frac{1}{L}\partial_x[(1+\frac{L_p}{L}+2L_pC_J\ddot{\Phi}'g_3)\Phi' + (g_3+3\frac{L_p}{L}g_3(1+g_4)+3L_pC_Jg_4)\Phi'^2$$
$$+(g_4+\frac{L_p}{L}(2g_3^2+g_4))\Phi'^3 + 2\frac{L_p}{L}g_3g_4\Phi'^4 + \frac{L_p}{L}g_4^2\Phi'^5]$$

Which can be rewritten as

$$\partial_x \partial_{\Phi'} \mathcal{L} = -\frac{1}{L_{\text{eff}}}\Phi'' - \frac{1}{L}\partial_x[(2L_pC_Jg_3\ddot{\Phi}')\Phi' + \gamma_3\Phi'^2 + \gamma_4\Phi'^3 + \gamma_5\Phi'^4 + \gamma_6\Phi'^5]$$

for effective non-linear coefficients

$$\gamma_3 = (1+\frac{3L_p(1+g_4)}{L}+\frac{3L_pC_Jg_4}{g_3})g_3$$
$$\gamma_4 = (1+\frac{L_p(2g_3^2/g_4+1)}{L})g_4$$
$$\gamma_5 = 2\frac{L_p}{L}g_3g_4$$
$$\gamma_6 = \frac{L_p}{L}g_4^2.$$

The Lagrangian is then

$$C_g\ddot{\Phi} - C_J\ddot{\Phi}'' - \frac{1}{L_{\text{eff}}}\Phi'' = \mathcal{N}' \tag{50}$$

where

$$\mathcal{N}' = \sum_{i=1}^{5}\mathcal{N}_i' \tag{51}$$

for

$$\mathcal{N}_1' = -\frac{(2L_pC_Jg_3)}{L}\partial_x(\ddot{\Phi}'\Phi') = -\frac{(2L_pC_Jg_3)}{L}(\ddot{\Phi}\Phi' + \ddot{\Phi}'\Phi'')$$
$$\mathcal{N}_2' = -\frac{2}{L}\gamma_3\Phi'\Phi''$$
$$\mathcal{N}_3' = -\frac{3}{L}\gamma_4\Phi'^2\Phi''$$
$$\mathcal{N}_4' = -\frac{4}{L}\gamma_5\Phi'^3\Phi''$$
$$\mathcal{N}_5' = -\frac{5}{L}\gamma_6\Phi'^4\Phi'']$$

The linear components undergo the same treatment, so we have for the linear phase mismatch

$$\Delta k_L = 2k_p - k_s - k_i$$

for the same (parasitic) definition of k_m as before.

If we make the same assumptions as before, that is

1. SVEA: $|A'_m| \ll |k_m A_m|$ therefore $\phi'_m \approx k_m \phi$, and same for the second derivative.
2. Small signal and idler amplitude: $|A_{s,i}|^2 \ll |A_p|$.
3. Relatively small parasitic inductance $L_p/L^2 << 1$.

then assumption 3 allows us to ignore $\mathcal{N}'_1$, $\mathcal{N}'_4$ and $\mathcal{N}'_5$ as they are proportional to L_p/L^2. Additionally, this assumption reduces the remaining nonlinear coefficient to simple forms:

$$\gamma_4 \to g_4,$$

$$\gamma_3 \to g_3 + g_{3,p}$$

.

$$\therefore \mathcal{N}' \to -\frac{2}{L}(g_3 + g_{3,p})\Phi'\Phi'' - \frac{3}{L} g_4 \Phi'^2 \Phi'' \tag{52}$$

Substitution of the ansatz into the propagation equation results in treatment identical to that of Ref. 1, with a small deviation of the 3WM coefficient.

2 Pump Depletion & Saturation Point

We consider a degenerate 4WM ansatz as before, but now aim to solve the coupled propagation equations without neglecting terms proportional to signal and idler amplitude. This should result is a set of coupled equations that deplete the pump when it's amplitude is near that of the signal tone, giving exact results for input power saturation. We begin by assuming perfect 4WM ($g_3 = 0$). The nonlinear component we're concerned with is given by

$$\frac{1}{L}\partial_x \mathcal{N}_3(\phi) = 3g_{4,p}\Phi'^2\Phi'' = \frac{3g_{4,p}}{8} \sum_{m,n,l\in\{p,s,i\}} (ik_m A_m \tilde{\phi}_m + c.c)(ik_n A_n \tilde{\phi}_n + c.c)(-k_l^2 A_l \tilde{\phi}_l + c.c) \tag{53}$$

where tilde now represents the normalized propagating wave function $\tilde{\phi} = \phi/A$. We form a set of differential equations by equating this sum to the reduced form of Eq. 25, given by 28:

$$\sum_{m\in\{p,s,i\}} -\frac{iC_g \omega_m^2}{k_m} \tilde{\phi}_m A'_m = \frac{1}{L}\partial_x \mathcal{N}_3(\phi). \tag{54}$$

To extract the relevant equations of motion for a particular signal, we multiply both sides by the complex conjugate of the selected wave and integrate over time, eliminating the terms that don't correspond to the correct frequency. For example, to extract the pump term, the selected wave has frequency ω_p.

$$-\frac{iC_g\omega_p^2}{k_p} A'_p = \int dt \tilde{\phi}_p^* \left(\frac{1}{L}\partial_x \mathcal{N}_3(\phi)\right) = \frac{3g_{4,p}}{8} \int dt \tilde{\phi}_p^* \sum_{m,n,l\in\{p,s,i\}} (ik_m A_m \tilde{\phi}_m + c.c)(ik_n A_n \tilde{\phi}_n + c.c)(-k_l^2 A_l \tilde{\phi}_l + c.c)$$

$$= \frac{3g_{4,p}}{8} \sum_{m,n,l\in\{p,s,i\}} k_m k_n k_l^2 \int dt (A_m \tilde{\phi}_{m-p} - A_m^* \tilde{\phi}_{-m-p})(A_n \tilde{\phi}_n - A_n^* \tilde{\phi}_{-n})(A_l \tilde{\phi}_l + A_l^* \tilde{\phi}_{-l})$$

By orthonormality, the selected wave extracts all combinations of m, n, and l that produce waves with the selected frequency ω_p. The product of two wave functions gives another wave at a frequency equal to the sum of the input waves as indicated by the subscript:

$$\phi_i \phi_j = \phi_{i+j}.$$

Therefore, finding the remaining terms can be framed as a search over linear combinations of n, m, and l that equal the selected wave's frequency. This becomes apparent after expanding the product in the integral:

$$\int dt(A_m\tilde{\phi}_{m-p} - A_m^*\tilde{\phi}_{-m-p})(A_n\tilde{\phi}_n - A_n^*\tilde{\phi}_{-n})(A_l\tilde{\phi}_l + A_l^*\tilde{\phi}_{-l})$$

$$= \int dt(A_mA_nA_l\tilde{\phi}_{m-p+n+l} - A_mA_n^*A_l\tilde{\phi}_{m-p-n+l} + A_mA_nA_l^*\tilde{\phi}_{m-p+n-l} - A_mA_n^*A_l^*\tilde{\phi}_{m-p-n-l}$$
$$-A_m^*A_nA_l\tilde{\phi}_{-m-p+n+l} + A_m^*A_n^*A_l\tilde{\phi}_{-m-p-n+l} - A_m^*A_nA_l^*\tilde{\phi}_{-m-p+n-l} + A_m^*A_n^*A_l^*\tilde{\phi}_{-m-p-n-l})$$

In the case $n = l$ (identical to the $m = l$ case)

$$= \int dt(A_mA_l^2\tilde{\phi}_{m-p+2l} - A_m|A_l|^2\tilde{\phi}_{m-p} + A_m|A_l|^2\tilde{\phi}_{m-p} - A_mA_l^{*2}\tilde{\phi}_{m-p-2l}$$
$$-A_m^*A_l^2\tilde{\phi}_{-m-p+2l} + A_m^*|A_l^*|^2\tilde{\phi}_{-m-p} - A_m^*|A_l|^2\tilde{\phi}_{-m-p} + A_m^*A_l^{*2}\tilde{\phi}_{-m-p-2l})$$
$$= \int dt(A_mA_l^2\tilde{\phi}_{m-p+2l} - A_mA_l^{*2}\tilde{\phi}_{m-p-2l} - A_m^*A_l^2\tilde{\phi}_{-m-p+2l} + A_m^*A_l^{*2}\tilde{\phi}_{-m-p-2l})$$

If $m = p$, the first, second and last terms die, while the third term lives only if $l = p$. The only nonzero selection $n = l = m = p$ gives

$$-k_p^4 A_p|A_p|^2.$$

In the case $m = n$

$$= \int dt(A_n^2A_l\tilde{\phi}_{-p+2n+l} - |A_n|^2A_l\tilde{\phi}_{-p+l} + A_n^2A_l^*\tilde{\phi}_{-p+2n-l} - |A_n|^2A_l^*\tilde{\phi}_{-p-l}$$
$$-|A_n|^2A_l\tilde{\phi}_{-p+l} + A_n^{*2}A_l\tilde{\phi}_{-p-2n+l} - |A_n|^2A_l^*\tilde{\phi}_{-p-l} + A_n^{*2}A_l^*\tilde{\phi}_{-p-2n-l})$$
$$= \int dt(A_n^2A_l\tilde{\phi}_{-p+2n+l} - 2|A_n|^2A_l\tilde{\phi}_{-p+l} + A_n^2A_l^*\tilde{\phi}_{-p+2n-l} - 2|A_n|^2A_l^*\tilde{\phi}_{-p-l}$$
$$+A_n^{*2}A_l\tilde{\phi}_{-p-2n+l} + A_n^{*2}A_l^*\tilde{\phi}_{-p-2n-l})$$

Which has two nonzero terms when $l = p$ and $m = n = s$ or $m = n = i$, corresponding to

$$-2k_p^2k_s^2A_p|A_s|^2$$

and

$$-2k_p^2k_i^2A_p|A_i|^2,$$

respectively. Finally, in the case $m \neq n \neq l$ there are three possible nonzero solutions that leverage energy conservation $2\omega_p = \omega_s + \omega_i$. Take first $m = p$, $n = s$, $l = i$:

$$k_p k_s k_i^2 \int dt (A_p A_s A_i \tilde{\phi}_{s+i} - A_p A_s^* A_i \tilde{\phi}_{-s+i} + A_p A_s A_i^* \tilde{\phi}_{s-i} - A_p A_s^* A_i^* \tilde{\phi}_{-s-i}$$
$$-A_p^* A_s A_i \tilde{\phi}_{-2p+s+i} + A_p^* A_s^* A_i \tilde{\phi}_{-2p-s+i} - A_p^* A_s A_i^* \tilde{\phi}_{-2p+s-i} + A_p^* A_s^* A_i^* \tilde{\phi}_{-2p-s-i})$$
$$= \int dt - A_p^* A_s A_i \tilde{\phi}_{-(s+i)+s+i} + A_p^* A_s^* A_i \tilde{\phi}_{-(s+i)-s+i} - A_p^* A_s A_i^* \tilde{\phi}_{-(s+i)+s-i} + A_p^* A_s^* A_i^* \tilde{\phi}_{-(s+i)-s-i}$$
$$= -k_p k_s k_i^2 A_p^* A_s A_i.$$

Then consider the permutations of m, n, and l which equivalently satisfy this condition with different wave vector coefficients:

$$-k_i(k_s k_i k_p) A_p^* A_s A_i, \quad (m,n,l) = (p,s,i), (s,p,i)$$
$$-k_s(k_s k_i k_p) A_p^* A_s A_i \quad (m,n,l) = (p,i,s), (i,p,s),$$
$$k_p(k_s k_i k_p) A_p^* A_s A_i \quad (m,n,l) = (s,i,p), (i,s,p)$$

Unlike the other index selections, these non-zero terms are left with a spacial term

$$e^{k_s + k_i - 2k_p} \equiv e^{\Delta k_L}$$

In summary, the indices which give non-zero terms are

$$m = n = l = p$$
$$m = n = s, \quad l = p$$
$$m = n = i, \quad l = p$$
$$m \neq n \neq l, \quad m, n, l \in \{p, s, i\}$$

The pump propagation equation is therefore

$$-\frac{iC_g \omega_p^2}{k_p} A_p' = \frac{3g_{4,p}}{8} \left[-k_p^4 A_p |A_p|^2 - 2k_p^2 k_s^2 A_p |A_s|^2 - 2k_p k_i^2 A_p |A_i|^2 \right.$$
$$-2k_p k_s k_i^2 A_p^* A_s A_i - 2k_p k_i k_s^2 A_p^* A_s A_i + 2k_s k_i k_p^2 A_p^* A_s A_i$$

$$A_p' = -\frac{i3k_p^2 g_{4,p}}{8C_g \omega_p} \left[k_p(k_p^2 |A_p|^2 + 2k_s^2 |A_s|^2 + 2k_i^2 |A_i|^2) A_p + 2k_s k_i (k_i + k_s - k_p) A_p^* A_s A_i e^{i\Delta k_L z} \right] \quad (55)$$

3 References

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