Efficient Sampling Algorithm for Multi-Qudit Clifford Circuits

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We present an algorithm for sampling Clifford circuits on n d-level quantum systems. Our formalism combines the conjugation tableau framework typically found in stabilizer code theory with a visualization that simplifies implementation and an $\mathcal{O}(n^2)$ runtime. Additionally, we provide several related results such as expressing the n-qudit Clifford generators in terms of elementary matrices and an analysis of the algorithm's overhead as a function of n and d.

I. INTRODUCTION

Qudits, $d \geq 2$ level quantum systems, have emerged as a powerful tool for storing and manipulating quantum information [1–3]. Due to their higher-dimensional Hilbert space, qudits can store exponentially more information than their binary counterparts, making them increasingly important in applied quantum information science. As hardware implementations of qudits become more common, the need for appropriate benchmarking procedures grows increasingly urgent.

Clifford circuits form a foundational class of quantum circuits that classical computers can efficiently simulate. The Clifford group constitutes a 3-design, effectively approximating Haar-random unitaries. This property makes Clifford circuits essential to quantum computing benchmarks, particularly in methods like Randomized Benchmarking. However, characterization procedures for qudit systems remain vastly unexplored compared qubit systems. In particular, qudit Randomized Benchmarking has, to the best of our knowledge, only been employed on single-qudit superconducting systems [4–6].

The naïve approach to sampling Clifford operations would be to generate all possible unitaries and then randomly select an index within the range of the group size. The challenge lies in sampling from a set that grows exponentially with the number of qudits and polynomially with their dimension. For example, while there are just 24 single-qubit Cliffords, two qubits support 11,520 Cliffords and three support 92,897,280. Similarly, while single-qutrit systems have 480 Cliffords, two qutrits have over four million.

State-of-the-art algorithms for randomly generating an n-qubit Clifford unitary achieve $\mathcal{O}(n^2)$ time complexity. Notably, Van Den Berg [7] presents an elegant derivation using conjugation tableaus, a formalism commonly found in the stabilizer literature from which the Clifford group gained its popularity. In this paper, we extend this to n-qudits using the finite-dimensional stabilizer formalism employed in [8] along with our own mathematical results.

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II. PRELIMINARIES

Throughout this work, our Hilbert space will be of constant prime dimension d with orthonormal basis $\{|q\rangle\}_{q\in\mathbb{Z}_d}$ where $\mathbb{Z}_d$ is the field of integers modulo d. We begin by introducing the generalized Pauli matrices known as the clock and shift operators X and Z:

$$X = \sum_{q=0}^{d-1} |q\rangle\langle q+1| \qquad Z = \sum_{q=0}^{d-1} \tau^{q^2} |q\rangle\langle q| \tag{1}$$

where addition is modulo d and $\tau = (-1)^d e^{i\pi/d}$. The order of τ for primes < 2 is d.

A. Weyl Operators

For two integers, $p, q \in \mathbb{Z}_d$, we define the single-qudit Weyl operator as

$$W_{p,q} = \tau^{pq} X^p Z^q \tag{2}$$

Beyond a single-qudit, we define vectors

$$\mathbf{p}, \mathbf{q} \in \mathbb{Z}_d^n \tag{3}$$

and the generalized Weyl operator may be expressed as

$$\begin{aligned} W_{\mathbf{p},\mathbf{q}} &= W_{p_0,q_0} \otimes \cdots \otimes W_{p_{n-1},q_{n-1}} \\ &= \tau^{\mathbf{p}\cdot\mathbf{q}} \left(X^{p_0} Z^{q_0} \otimes \cdots \otimes X^{p_{n-1}} Z^{q_{n-1}}\right) \end{aligned} \tag{4}$$

For compactness sake we concatenate $\mathbf{p}$ and $\mathbf{q}$ into a single vector $\mathbf{v} = (\mathbf{p}, \mathbf{q}) \in \mathbb{Z}_d^{2n}$ such that $W_{\mathbf{p},\mathbf{q}} \equiv W_{\mathbf{v}}$. One can show the following convenient properties:

1. $W_{\mathbf{u}} W_{\mathbf{v}} = \tau^{-[\mathbf{u},\mathbf{v}]} W_{\mathbf{u}+\mathbf{v}}$
2. $W_{\mathbf{v}}^\dagger = W_{-\mathbf{v} \bmod d}$
3. $W_{\mathbf{u}}$ and $W_{\mathbf{v}}$ commute iff $[\mathbf{u},\mathbf{v}] = 0 \bmod d$

where $[\mathbf{u},\mathbf{v}]$ denotes the symplectic inner product:

$$[\mathbf{u},\mathbf{v}] \equiv \mathbf{u}^\top \sigma \mathbf{v}, \quad \text{where} \quad \sigma = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}. \tag{5}$$

We also write $[\mathbf{u},\mathbf{v}]_d = [\mathbf{u},\mathbf{v}] \pmod d$. Notice that $[\mathbf{v},\mathbf{v}] = 0$ and $[\mathbf{u},\mathbf{v}] = -[\mathbf{v},\mathbf{u}]$. Later, we show a one-to-one relationship between nonzero values of the symplectic inner product and the commutator relation of the Weyl operators (See Section II C 2).

B. Pauli Vectors

Generalized Pauli matrices are proportional to Weyl operators by a factor of τ^2. Therefore,

$$P_{\phi,\mathbf{v}} = \tau^{-2\phi} W_{\mathbf{v}} \tag{6}$$

for $\phi \in \mathbb{Z}_d$ and $\mathbf{v} \in \mathbb{Z}_d^{2n}$ as before. The Weyl operator formalism allows us to speak of n-qudit Paulis in terms of characteristic vectors $\bar{\mathbf{v}} \in \mathbb{Z}_d^{2n+1}$ called *Pauli vectors.* We write the Pauli vector by embedding the parameters into a single two-block vector:

$$\bar{\mathbf{v}} = [\ \phi \mid \mathbf{v}\ ]^\top \tag{7}$$

which is written as a column vector so we may perform matrix multiplication from the left. We identify the generators of the Pauli group, $\{\tau^{-2}I, X, Z\}$, when $\bar{\mathbf{v}}$ is a standard basis vector $\hat{\mathbf{e}}_j$ where $j \in \{0 \cdots 2n\}$, and call these the Pauli bases. All choices of Pauli vectors then form the n-qudit Paulis $\mathcal{P}_n^d$:

$$\mathcal{P}_n^d = \left\{ P_{\bar{\mathbf{v}}} \;\middle|\; \bar{\mathbf{v}} \in \mathbb{Z}_d^{2n+1} \right\} \tag{8}$$

C. The Clifford Group

The n-qudit Clifford group is defined as the normalizer of the Pauli group up to a global phase:

$$\mathcal{C}_n^d = \left\{ U \in U(d^n) \;\middle|\; P \in \mathcal{P}_n^d/\{I\},\quad U(P) \in \mathcal{P}_n^d \right\} /U(1). \tag{9}$$

where parenthesis implies conjugation $U(M) = UMU^\dagger$. The generators are the Fourier gate (a $d > 2$ extension of the Hadamard gate) F, the phase gate S, the multiplication gate M_a for a in the multiplicative group of $\mathbb{Z}_d$, and the two-qudit controlled-X gate, CX [8]:

$$F = \frac{1}{\sqrt{d}} \sum_{p,q=0}^{d-1} \tau^{2pq} |p\rangle\langle q| \tag{10a}$$

$$S = \sum_{q=0}^{d-1} \tau^{q^2} |q\rangle\langle q| \tag{10b}$$

$$M_a = \sum_{q=0}^{d-1} |aq\rangle\langle q| \pmod d \tag{10c}$$

$$CX = \sum_{q_c=0}^{d-1} \sum_{q_t=0}^{d-1} |q_c\rangle\langle q_c| \otimes |q_t + q_c\rangle\langle q_t| \pmod d \tag{10d}$$

1. Actions on the Pauli Group

Denote X_i (Z_i) the Pauli which acts only X (Z) on the qudit at index $i \in \mathbb{Z}_n$:

$$X_i = I^{\otimes i} \otimes X \otimes I^{\otimes n-1-i} \tag{11}$$

$$Z_i = I^{\otimes i} \otimes Z \otimes I^{\otimes n-1-i}. \tag{12}$$

Then, for some $C \in \mathcal{C}_n^d$,

$$C(X_i) \mapsto P,\quad C(Z_i) \mapsto Q, \quad P, Q \in \mathcal{P}_n^d \tag{13}$$

The operator $C^\dagger$ which maps P and Q back to the basis Paulis is also a member of Clifford group.

The first stage of the algorithm presented herein is a sampling procedure that is conditional upon the commutation relation between two randomly selected Paulis, P and Q. Next we examine how the commutation relation of these Paulis determines whether or not there exists a Clifford such that

$$C^\dagger(P) \mapsto X_i\ ,\quad C^\dagger(Q) \mapsto Z_i. \tag{14}$$

2. Commutation Relations

It is convenient that the commutation relations of Pauli operators extend to Paulis under conjugation. Defining the commutator as

$$[A, B] = ABA^\dagger B^\dagger \tag{15}$$

we relate the commutation relation of distinct Pauli operators to the symplectic inner product of their vectors [9]. For $P_j \propto W_{\mathbf{v}_j}$, $P_h \propto W_{\mathbf{v}_h}$,

$$[P_j, P_h] = \tau^{2[\mathbf{v}_j, \mathbf{v}_h]_d} I. \tag{16}$$

A pair of Pauli operators can fail to commute in $d-1$ ways, each corresponding to a value of $[\mathbf{v}_j, \mathbf{v}_h]_d \in \mathbb{Z}_d$ and each with equal multiplicity. Defining the conjugation adjoint as $C(P)^\dagger = CP^\dagger C^\dagger = C(P^\dagger)$ it is easy to show that Cliffords preserve the commutation relation of the Pauli operators:

$$[C(P_j), C(P_h)] = [P_j, P_h]. \tag{17}$$

For two pairs of distinct Paulis, there exists a Clifford that maps one pair to the other only if the commutation relation is preserved.

3. Symplectic Clifford Operators

We define the symplectic Clifford operators as those which map basis Paulis to other Weyl operators:

$$\sigma\mathcal{C}_n^d = \left\{ C \in \mathcal{C}_n^d \;\middle|\; C(W_{\hat{\mathbf{e}}_j}) \in \mathcal{W}\ \ \forall\ \ 0 \le j \le 2n \right\} \subseteq \mathcal{C}_n^d \tag{18}$$

All Paulis may be written as products of Weyl operators with basis vector arguments, therefore a Clifford operator is uniquely characterized by it's transformation of these basis vectors.

We define a *symplectic transformation* as an operator $\mathcal{S}$ which preserves the symplectic inner product under conjugation:

$$\mathcal{S} :\ \mathbb{Z}_d^{2n} \mapsto \mathbb{Z}_d^{2n} \quad \text{s.t.} \quad \mathcal{S}^\top \sigma \mathcal{S} = \sigma \tag{19}$$

The group of such transformations is called $\mathrm{Sp}_{2n}(\mathbb{Z}_d)$. We invoke Theorem 5 from [8] which tells us that for any $C \in \sigma\mathcal{C}_n^d$, there exists $\mathcal{S} \in \mathrm{Sp}_{2n}(\mathbb{Z}_d)$ such that

$$C(W_{\mathbf{v}}) = W_{\mathcal{S}\mathbf{v}} \;\; \forall \;\; \mathbf{v} \in \mathbb{Z}_d^{2n}. \tag{20}$$

Elements of the symplectic Clifford group simultaneously preserve the commutation relation of Pauli operators and the symplectic inner product of Pauli vectors. Consider two two distinct Pauli vectors under symplectic transformation associated with a symplectic Clifford operator, $C(W_{\hat{\mathbf{e}}_j}) = W_{\mathbf{v}}$ and $C(W_{\hat{\mathbf{e}}_h}) = W_{\mathbf{w}}$, then

$$[\mathbf{v}, \mathbf{w}] = \hat{\mathbf{e}}_j^\top \mathcal{S}^\top \sigma \mathcal{S} \hat{\mathbf{e}}_h = \hat{\mathbf{e}}_j^\top \sigma \hat{\mathbf{e}}_h = [\hat{\mathbf{e}}_j, \hat{\mathbf{e}}_h] \pmod d \tag{21}$$

$$= \sum_{i=1}^{n} \delta_{j,i}\ \delta_{h,n+i} - \sum_{i=1}^{n} \delta_{j,n+i}\ \delta_{h,i} = \begin{cases} 1 & \text{if } h = j+n \\ -1 & \text{if } h = j-n \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

Moreover, this relation holds for the adjoint of such an operator. Importantly, if $[\mathbf{v}, \mathbf{w}] = \pm 1$ there exists a Clifford such that $C(W_{\mathbf{v}}) = W_{\hat{\mathbf{e}}_j}$ and $C(W_{\mathbf{w}}) = W_{\hat{\mathbf{e}}_h}$.

III. THE TABLEAU FORMALISM

For n qudits, a $2n \times 2n+1$ *tableau* is used to characterize the action of an arbitrary Clifford $C \in \mathcal{C}_n^d$. Denote the tableau $\mathcal{T}(C)$, with elements $\mathcal{T}_{i,j} \in \mathbb{Z}_d$ for $i, j \in \{0 \cdots n-1\}$. Mathematically, the tableau represents a matrix with transposed Pauli vectors as its columns:

$$\mathcal{T}(C) = \left(\begin{array}{c|c} \phi_0 & \mathbf{v}_0 \\ \phi_1 & \mathbf{v}_1 \\ \vdots & \vdots \\ \phi_{2n-1} & \mathbf{v}_{2n-1} \end{array} \right)^\top = \left(\vec{\Phi} \,\middle|\, \mathbf{W} \right)^\top = \left(\vec{\Phi} \,\middle|\, \mathbf{X} \,\middle|\, \mathbf{Z} \right)^\top \tag{23}$$

We transpose the matrix in the above expression to help the reader transition to the visual representation presented momentarily. The Pauli vector of column j reveals the image $CW_{\hat{\mathbf{e}}_j}C^\dagger = W_{\mathbf{v}_j}$. Conversely, the resulting Weyl acting on qudit i is characterized by the phase in the first row and the power of X (Z) found in row i ($i+n$), which lives in the so-called X (Z) block.

A. Symplectic Transformations

Each of the Clifford operators maps Weyls to Weyls and therefore may be represented by symplectic operators $\mathcal{S} \in \mathrm{Sp}_{2n}(\mathbb{Z}_d)$ acting on qudits with indices $0 \leq i, h \leq n-1$. For clarity, we decompose the symplectic operations given in Ref.[8] into products of elementary matrices:

$$\mathcal{S}_S = L_{i+n,i}(1) \tag{24a}$$

$$\mathcal{S}_F = T_{i,i+n} D_{i+n}(-1) \tag{24b}$$

$$\mathcal{S}_{M_a} = D_i(a) D_{i+n}^{-1}(a) \tag{24c}$$

$$\mathcal{S}_{CX} = L_{i+n,h+n}(-1) L_{h,i}(1) \tag{24d}$$

where $L_{\alpha,\beta}(\gamma)$ is a transvection (or shear-mapping) which adds γ times row β to row α; $T_{\alpha,\beta}$ is a permutation matrix which swaps rows α and β; and $D_\alpha(\gamma)$ is the scalar multiplication of row α by γ. Using this decomposition and the respective properties of the elementary matrices, it is easy to derive valuable operations using the Clifford generators, such as the $SWAP$ gate:

$$SWAP_{i,h} = F_i^2 CX_{i,h} CX_{h,i}^\dagger CX_{i,h} \;:\; |\psi_i\rangle|\psi_h\rangle \mapsto |\psi_h\rangle|\psi_i\rangle \tag{25}$$

The action of a Clifford operator on a tableau is expressed by left multiplication with the respective symplectic matrix:

$$\mathcal{S}_U \mathcal{T}(C) = \left(\vec{\Phi} \,\middle|\, U \mathbf{W} U^\dagger \right)^\top \tag{26}$$

From our decomposition of the symplectic operators into elementary matrices, we can derive the following properties (see Fig. 1):

1. **Phase Gate:** When acting on qudit i, this gate adds the X block Pauli vector entries to the corresponding Z block.

2. **Fourier Gate:** When acting on qudit i, this gate multiplies row $i+n$ in the Z block by -1, then swaps it with row i of the X block.

3. **Multiplicative Gate:** This gate scales row i of the X block by its parameter a and row $i+n$ by a^{-1}.

4. **Controlled-X Gate:** With control qudit i and target qudit h, this gate adds the control column of the X block to the target column of the X block, while subtracting the target column of the Z block from the control column of the Z block.

B. Visual Representation

Next, we turn to a visual representation of the tableau which provides an intuitive way to understand the algorithm proposed in this paper. Beginning with the definition in Eq. 23, we ignore the transpose then perform a reindexing of the rows such that for qudit i, adjacent rows correspond to the Pauli vectors associated with the images $C(W_{\hat{\mathbf{e}}_i}) \equiv C(X_i) = W_{\mathbf{v}_x}$ and $C(W_{\hat{\mathbf{e}}_{i+n}}) \equiv C(Z_i) = W_{\mathbf{v}_z}$, respectively (see Fig. 2a). The exact index map interleaves the rows and is given by

$$i \mapsto \tilde{i} = 2i \;\; (i < n)$$

$$i \mapsto \tilde{i} = 2(i-n)+1 \;\; (i \geq n).$$

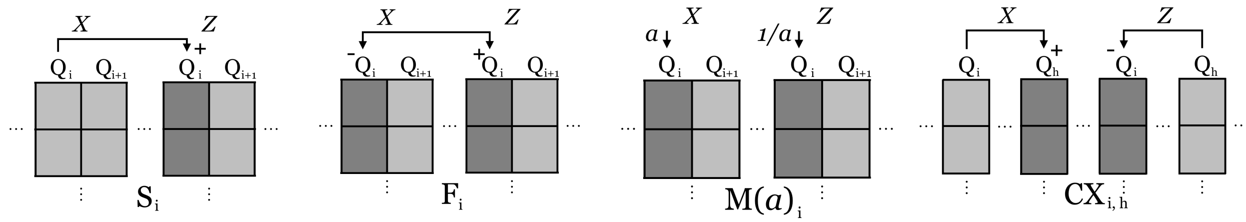

FIG. 1: In the visual representation of the tableau, the symplectic matrix corresponding to each Clifford generator applies a unique transformation to the tableau columns.

IV. ALGORITHM FOR GENERATING n-QUDIT CLIFFORD CIRCUITS

Following the tableau formalism presented in Ref [7], we present an algorithm that samples random elements of $\mathcal{C}_n^d$. The algorithm initializes $\tilde{i} = 0$ and iterates while $\tilde{i} < n$:

1. Randomly sample two Pauli vectors $\mathbf{v}_x$ and $\mathbf{v}_z$ $\in \{0\}^{2\tilde{i}} \oplus \mathbb{Z}_d^{2(n-\tilde{i})}$ that satisfy the commutation relation of the basis operators X and Z. This is required by the preservation of commutation relations under the Clifford action (see Section IV A).

2. Algorithmically sweep the populated tableau to arrive at the Pauli basis vectors $\hat{\mathbf{e}}_{\tilde{i}}$, $\hat{\mathbf{e}}_{\tilde{i}+1}$, therefore finding the inverse map $C^\dagger$ such that

$$C^\dagger W_{\mathbf{v}_x} C \mapsto X_i \tag{27a}$$
$$C^\dagger W_{\mathbf{v}_z} C \mapsto Z_i. \tag{27b}$$

3. Increment $\tilde{i}$ by 1: $\tilde{i} \to \tilde{i} + 1$. The algorithm repeats steps 1 and 2 for each qudit.

A. Sampling Paulis

Conjugation under symplectic Clifford operators preserves the symplectic inner product and thus commutation relations among Weyl operators. Therefore, if $C \in \sigma\mathcal{C}_n^d$, then by Eq.22 the rows of $\mathcal{T}(C)$ are orthogonal with respect to the symplectic inner product, *except* for rows h and $h \pm n$, which have symplectic inner product of ± 1. Before reindexing, these rows correspond to the images of the shift and clock operators under C. After reindexing, these become adjacent rows h and $h+1$.

The sampling procedure is as follows:

1. Sample two phase values $\phi_{\tilde{i}}, \phi_{\tilde{i}+1} \in \mathbb{Z}_d$. As these do not affect commutation relations, this can be done independently.

2. Initialize a random Pauli vector $\mathbf{v}_x \in \mathbb{Z}_d^{2n}$. If it is the identity, it will commute with the second Pauli vector; therefore, we discard it and sample again.

3. Initialize a second random Pauli vector $\mathbf{v}_z \in \mathbb{Z}_d^{2n}$ such that it does not commute with $\mathbf{v}_x$, i.e., $[\mathbf{v}_x, \mathbf{v}_z] = 1$.

As shown in [7], the probability of sampling two anti-commuting Pauli vectors for qubits is greater than 3/8. For n qudits, we sample $2n$ integers from $\mathbb{Z}_d$, yielding d^{2n} possible Pauli vectors. The first Pauli, P_1, must not be the identity, leaving $d^{2n} - 1$ valid Paulis. The number of Paulis which fail to commute with P_1 in the same manner as the shift and clock operators is d^{2n}/d. Therefore the probability of sampling a valuable Pauli pair is

$$\frac{(d^{2n}-1)d^{2n-1}}{(d^{2n})^2} = \frac{1}{d}\left(1 - \frac{1}{d^{2n}}\right) \geq \frac{1}{d} - \frac{1}{d^3}. \tag{28}$$

After completing this stage, we arrive at the populated tableau in Figure 2a.

B. Sweeping the Tableau

At the heart of the algorithm is a six-step sweeping procedure that acts on pairs of rows $r_1 \equiv 2\tilde{i}$ and $r_2 \equiv 2\tilde{i} + 1$.

1. Clear Z entries of row r_1 (Fig. 2b):

   (a) For every qudit of row r_1 at index $> \tilde{i}$, apply a Fourier gate if the entry in the X block is 0. Otherwise, if the corresponding entry of the Z block is nonzero, apply a phase gate.

2. Reduce the X entries of row r_1 to a basis vector $\hat{\mathbf{e}}_{2\tilde{i}}$, i.e., zeros everywhere except for a 1 at entry $\mathcal{T}_{2\tilde{i},2\tilde{i}}$ (Fig 2c):

   (a) Create a list of indices $\mathcal{J}$ corresponding to the nonzero entries in the X block of r_1.

   (b) While $|\mathcal{J}| > 1$ (i.e. $\mathcal{J}$ is not a singleton) apply CX gates on pairs of qudits corresponding to the entries in odd indices ($\leq |\mathcal{J}|$) of $\mathcal{J}$.

   (c) When an X block entry of target qudit cycles to 0, remove its index from $\mathcal{J}$ and repeat from step 2a.

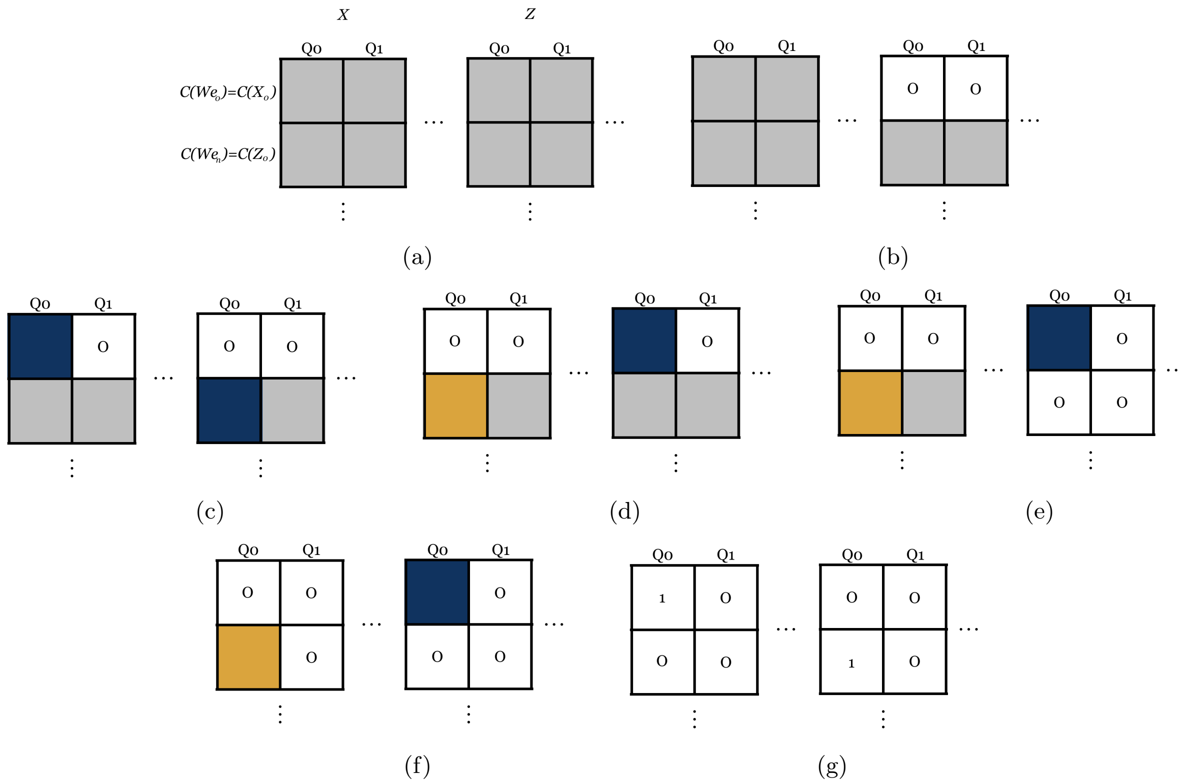

FIG. 2: The first two rows of a tableau for two qudits capture the signature of a Clifford operator C acting on the first qudit — specifically, the images $C(X_0)$ and $C(Z_0)$. Gray entries have values in $\mathbb{Z}_d$, while blue and gold entries have arbitrary nonzero values less than d.

3. If the nonzero entry is not in column $2\tilde{i}$, i.e., if $\mathcal{J}[0] \neq 2\tilde{i}$, perform a SWAP gate between column $\mathcal{J}[0]$ and column $2\tilde{i}$.

4. If the second row is already in the $Z_{2\tilde{i}}$-configuration (i.e. zeros except a 1 at entry $2\tilde{i}$ in the Z block), skip this step. Otherwise, perform a Fourier gate on qudit $2\tilde{i}$. Repeat steps 1 and 2 as before, but on the second row. Then perform an additional Fourier gate (see Fig. 2d, 2e, and 2f).

5. The first entry of r_1 and the nth entry of r_2 are now equal. If both equal 1 (which is required to satisfy the commutation relation of the basis operators), skip this step. Otherwise, apply a multiplicative gate with $a = r_2[n]$, the first entry of the $2\tilde{i}+1$ row of the Z block (see Fig. 2g).

6. Clear the phase vector by repeatedly applying clock and shift operators. While $\phi_{r_1} \neq 0$ or $\phi_{r_2} \neq 0$:

   (a) If both are nonzero, apply a $Y \propto XZ$ gate to qudit $2\tilde{i}$. This gate does not commute with either row, thereby decrementing the phase by 1.

   (b) If only ϕ_{r_1} is nonzero, apply a Z gate.

   (c) If only ϕ_{r_2} is nonzero, apply a X gate.

C. Efficacy

No Clifford operator maps two distinct Weyl operators to the same Weyl operator. This implies that the sample space of valid Pauli vectors is the same size as the set of Clifford operators acting on them. From Eq. 28, there are $d^{2n-1}(d^{2n}-1)$ valid Pauli vector pairs, ignoring phases. As a function of a subset of $m < n$ qudits, the multiplicity is given by

$$\Omega(m, d) = d^{2m-1}(d^{2m}-1)$$

. Consideration m to range from 0 to $n-1$, as in the iterative process of the algorithm, we maximize the Clifford sample space:

$$\Omega(n, d) = \prod_{i=0}^{n-1} \Omega(n-i, d) = |\mathrm{Sp}_{2n}(\mathbb{Z}_d)|. \tag{29}$$

We independently sample phases, which account for the total cardinality of the generalized Pauli group $|\mathcal{P}_n^d| = d^{2n}$. Thus, the cardinality of the accessible Clifford space is:

$$|\mathcal{C}_n^d| = |\mathcal{P}_n^d||\mathrm{Sp}_{2n}(\mathbb{Z}_d)| = d^{2n}d^{n^2}\prod_{i=1}^{n-1}(d^{2i}-1). \tag{30}$$

This expression agrees with well-known results in the field such as $\mathcal{C}_n^d/\mathcal{P}_n^d \cong \mathrm{Sp}_{2n}(\mathbb{Z}_d)$. In the case of n-qubits, this reduces to the known form $\mathcal{C}_n^2 = 2^{n^2+2n}\prod_{i=1}^{n-1}(4^i-1)$ [10] [11].

V. DISCUSSION

This paper presents an algorithm for sampling n-qudit Clifford circuits, which has a runtime complexity of $\mathcal{O}(n^2)$. By extending the conjugation tableau formalism to d-dimensional systems, we have provided a method that is both theoretically rigorous and practically implementable, aided by a visual representation. The algorithm achieves uniform sampling across the entire Clifford group as confirmed by our cardinality analysis, which is consisted with established results in the literature.

The efficient sampling technique has immediate applications in quantum characterization protocols for qudit systems, particularly in extending Randomized Benchmarking beyond qubits and single-qutrit implementations. Future work could focus on optimizing the algorithm for specific values of d, developing variants for common qudit dimensions, or exploring connections with fault-tolerant protocols for higher-dimensional systems. As qudit-based quantum computing hardware continues to develop, this sampling algorithm provides an accessible tool for comparing emerging quantum processors.

CODE IMPLEMENTATION

The algorithm presented in this paper has been implemented and is publicly available. The code can be accessed through the following GitHub repository: https://github.com/qnl/qnlib. This repository contains the full implementation of the algorithm, along with documentation and examples to facilitate its use in quantum circuit sampling and related tasks.

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