Introduction

In our experiment, we are interested in having full control of the interaction between neighboring qutrits via a tunable resonator. Tuning of the resonator is controlled by biasing a flux line, which has mutual inductance with the tunable resonator's SQUID. The interaction between these components is known to provide a decoherence channel. In this document, we investigate several decoherence effects. Particularly, we are concerned with the decoherence and dephasing times that result from tuning our coupler with the flux line.

Decoherence can by described as noise in the Hamiltonian's external parameters, λ_i. Generally, these are fluctuations in charge, magnetic flux, and critical current. Although other contributions to noise are thought to be present, the work surrounding these sources of decoherence is less important for our experiment so for example, we will not consider dielectrics losses here.

Generally, the Hamiltonian of a qutrit is of the form

$$\hat{H} = \vec{h}(\lambda_i)\vec{\sigma} \tag{1}$$

where $\vec{\sigma}$ contains the three spin-1 Pauli matrices, and $\vec{h}$ contains their respective coefficients. The noise in our parameters may be expressed as

$$\lambda_i = \lambda_i^o + \delta\lambda_i \tag{2}$$

where λ_i^o is the dc component of the parameter, and $\delta\lambda_i$ is the fluctuating noise term. The noise leads to two decoherence effects:

i) For sufficiently low frequencies, fluctuations in our parameters may be treated with the adiabatic approximation, leading to random shifts in the transition frequency of the qutrit. This gives a pure dephasing on the time scale T_2.

ii) Higher frequency fluctuations break the adiabatic approximation and induce transitions between qutrit states. This is energy relaxation on the time scale T_1.

Estimates for Qutrit Relaxation Time (T_1)

The 2007 paper by Koch et al. (https://doi.org/10.1103/PhysRevA.76.042319) covers a cohesive summary of the transmon decoherence channels which allow estimates for T_1 times. However, these channels are for TLSs, not qutrit systems like those we are dealing with. As such, we extend these concepts to three-level systems in this section.

Spontaneous Emission

We can get a decent approximating for the relaxation time due to the spontaneous emission of photons from our system by treating it as a Hertzian dipole with wavevector $k = \frac{\omega_r}{c}$ and dipole moment $\mathbf{p}$. The total power radiated by such a system (see Chapter 9.2 of Classical Electrodynamics by Jackson) is

$$P = \frac{ck^4}{3}|\mathbf{p}|^2 = \frac{1}{4\pi\varepsilon_0}\frac{p^2\omega_r^4}{3c^3}. \tag{3}$$

In our circuit, power is exerted by cooper pairs tuning through a barrier of length L, which is analogous to a center-fed linear antenna with nodes of length L, so for $p = 2eL$

$$T_1^{rad}|_{01} \equiv \frac{\hbar\omega_{01}}{P} = \frac{12\pi\varepsilon_0\hbar c^3}{d^2\omega_{01}^3} \tag{4}$$

and

$$T_1^{rad}|_{12} \equiv \frac{\hbar\omega_{12}}{P} = \frac{12\pi\varepsilon_0\hbar c^3}{d^2\omega_{12}^3} \tag{5}$$

Purcell Effect
A system in a resonating cavity has an altered spontaneous emission rate, this is known as the Purcell Effect. For the transmon coupled to a transmission line resonator, the same effect will occur and each transmon level will experience a different change to its spontaneous relaxation rate.
We may apply Fermi's Golden Rule to the Hamiltonian for the interaction of the resonator with its bath to estimate the relaxation rate. Take the bath Hamiltonian

$$H_B(\kappa) = \hbar\sum_k \lambda_k[\hat{b}_k^\dagger\hat{\alpha} + \hat{\alpha}^\dagger\hat{b}_k] \tag{6}$$

where $\hat{b}_k$ and $\hat{b}_k^\dagger$ are the bath operators for mode k. λ_k determines the coupling strength per mode. Now take the generalized Jayes-Cumming Hamiltonian, denoted H_S for Hamiltonian of the system,

$$H_S = \hbar\sum_j \omega_j|j\rangle\langle j| + \hbar\omega_r\hat{\alpha}^\dagger\hat{\alpha} + \hbar\sum_{i,j} g_{i,j}|i\rangle\langle j|(\hat{\alpha} + \hat{\alpha}^\dagger) \tag{7}$$

where

$$\hbar g_{ij} = 2\beta e V_{rms}^o\langle i|\hat{n}|j\rangle \tag{8}$$

gives the coupling energies between states. The various coefficients are defined as

Variable	Definition
Transmon Characteristic Frequency (ω_r)	$1/L_rC_r$
Ratio of gate to total capacitance (β)	C_g/C_Σ
Root-mean-squared voltage of the local oscillator (V_{rms}^o)	$\sqrt{\hbar\omega_r/2C_r}$
Quantum Number Operator ($\hat{n}$)	$\hat{\alpha}^\dagger\hat{\alpha}$

After approximating non-adjacent coupling terms to 0 (in the transmon regime)

$$H_S \approx \hbar\sum_j \omega_j|j\rangle\langle j| + \hbar\omega_r\hat{\alpha}^\dagger\hat{\alpha} + [\hbar\sum_i g_{i,i+1}|i\rangle\langle i+1|\hat{\alpha}^\dagger + \text{h.c}]. \tag{9}$$

If we apply the interaction analysis covered in Chapter 3.3 of The Theory of Open Quantum Systems by H.-P. Breuer and F. Petruccione, we arrive at a relaxation rate defined as

$$\Gamma_{f,i}(\omega) \equiv \int_0^\infty ds e^{i\omega s}\langle B_\alpha^\dagger(t)B_\beta(t-s)\rangle = \frac{1}{2}\gamma(\omega)_{f,i} + iS_{f,i}(\omega) \tag{10}$$

where

$$\langle B_\alpha^\dagger(t)B_\beta(t-s)\rangle \equiv \text{tr}_B\{B_\alpha^\dagger(t)B_\beta(t-s)\rho_\beta\} \tag{11}$$

are the reservoir correlation functions and

$$B_\alpha(t) = e^{iH_Bt}B_\alpha e^{-iH_Bt} \tag{12}$$

are the interaction picture operators. Notice Γ is a complex tensor. The real component represents the physical dissipator, whereas the imaginary part gives the so-called lamb-shift or renormalization of the unperturbed energy levels induced by the system-reservoir coupling. Solving for the disspator terms in our system (as done by Koch et al) gives

$$\gamma_k^{(f,i)} = \frac{2\pi}{\hbar} p(\omega_k) |\langle 1_k, f| \hbar \sum_{k'} \lambda_{k'} [\hat{b}_{k'}^\dagger \hat{\alpha} + \hat{\alpha}^\dagger \hat{b}_{k'}] |0, i\rangle|^2 \tag{13}$$

This value is the rate that the transmon emits 1 photon with energy $\hbar\omega_k = E_i - E_f$ to the bath. The Reservoir's density of states at this energy is $p(\omega_k)$. If we define

$$\kappa = 2\pi\hbar p(\omega_k)|\lambda_k|^2 \tag{14}$$

then

$$\gamma_\kappa^{(f,i)} = \kappa |\langle f|\hat{\alpha}|i\rangle|^2 \tag{15}$$

In the absence of photons in the cavity the higher transmon levels will see a Purcell-induced relaxation rate

$$\gamma_\kappa^{(i,i+1)} = \kappa \frac{g_{i,i+1}^2}{(\omega_{i,i+1} - \omega_r)^2}. \tag{16}$$

This assumption is valid because the base temperature of about 16mK is close enough to 0 photon energy.

Capacitive (Drive) Coupling

Charge-coupling to a drive line introduces voltage noise into the system. We approximate the Johnson-Nyquist noise PSD as [1]

$$S(\omega)_V = \hbar\omega \text{Re}[Z(\omega)](1 + \coth \frac{\hbar\omega}{k_B T}). \tag{17}$$

The relaxation rate resulting from such noise is given by

$$\Gamma_1^{i \to j} = \frac{1}{\hbar} |\langle i|A|j\rangle|^2 S(\omega)_V \tag{18}$$

where $A = V_g \hat{n}$ for $V_g = \frac{E_c}{2} n_g$. The charge offset, $n_g = \frac{C_c V_c}{2e}$ where C_c is the capacitance to the drive line, and V_c is the root mean square voltage noise from the PSD above:

$$V_c = \int_0^\infty S(\omega)_v d\omega \tag{19}$$

Flux Coupling

Most important to our experiment is the relaxation due to the transmon coupler interacting with the flux line bias during control. We recognize two causes for this relaxation. These are, (i) intentional coupling between the SQUID loop and flux bias through mutual inductance, denoted M hereafter, which allows for E_J tuning, and (ii) the entire transmon couples to the flux bias via M' (see Figure 1)

I) Let the total flux thorugh the SQUID be the sum of the environmental flux from the control line and the noise flux from other external fields, $\Phi = \Phi_e + \Phi_n$, and assume $\Phi_e >> \Phi_n$. Taylor expansion of the Josephson Hamiltonian can be represented as

$$H_J \mapsto H_J + \Phi_n \hat{A} \tag{20}$$

where

$$\hat{A} = \frac{\partial H_J}{\partial \Phi}|_{\Phi_e} = E_{J\Sigma} \frac{\pi}{\Phi_o} [\sin(\pi\Phi_e/\Phi_o)\cos(\hat{\varphi}) - d\cos(\pi\Phi_e/\Phi_o)\sin(\hat{\varphi})] \tag{21}$$

[1] As a side note, the most accurate form of this PSD is $S(\omega)_V = \hbar\omega \text{Re}[Z(\omega)](1 + \coth \frac{\hbar\omega}{k_B T}) \frac{2}{e^{\hbar\omega/k_B T}}$ but the exponent is O(1) if $\hbar\omega \approx k_B T$

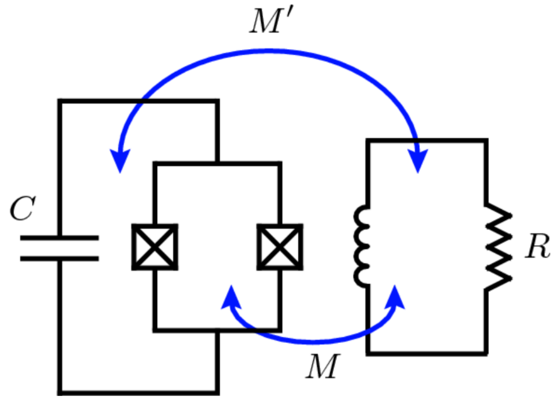

FIGURE 1. The transmon (left) experiences mutual inductances M, and M' with the flux line (right). (Image taken from Koch et al (2007))

for total Josephson Energy $E_{J\Sigma} = E_{J_1} + E_{J_2}$ and Junction asymmetry $d = (E_{J_1} - E_{J_2})/E_{J\Sigma}$. The term $\Phi_n \hat{A}$ in Eq. (12) captures the sensitivity of the Josephson Hamiltonian to flux, scaled by the perturbative flux noise.

Using perturbation theory, we may relate the relaxation rate to the power spectrum as

$$\Gamma_1^{f,i} = 1/T_1^{f,i} = \frac{1}{\hbar^2}|\langle f|\hat{A}|i\rangle|^2 S_{\Phi_n}(\omega_{f,i}) \tag{22}$$

where

$$S_{\Phi_n}(\omega_{f,i}) = \int_{-\infty}^{+\infty} d\tau e^{i\omega_{f,i}\tau}\langle \Phi_n(\tau)\Phi_n(0)\rangle = M^2 S_{I_n}(\omega_{f,i}). \tag{23}$$

is the mutual inductance relation that comes from the correlation function for the bath operator (see Appendix A). The flux bias line could be at a much higher effective noise temperature so we need to estimate the Johnson-Nyquist noise. The fluctuations in the voltage can be related to the total parallel impedance $Z_t(\omega)$ and the inductance through the fluctuation-dissipation theorem[2]:

$$S_{I_n}(\omega) = \frac{1}{\omega^2 L^2} I_p^2 S_{V_n} = \frac{\hbar\omega}{\omega^2 L^2} Re[Z_t(\omega)]\left[\coth\frac{\hbar\omega}{2k_B T} + 1\right]. \tag{24}$$

where T is the temperature of the bath and $I_p = \sqrt{\frac{C}{L}} = \frac{1}{\omega L}$ is the persistent current of the superconductor which depends on the control circuit inductance. At temperatures $k_B T << \hbar\omega_q$ and impedance of the flux line inductor $Z_L << 50\Omega$, the current quantum noise is approximated as

$$S_{I_n} \approx 2\Theta(\omega_{f,i})\hbar\omega_{f,i}/R, \tag{25}$$

where $\Theta(\omega)$ is the step function. We may represent the SQUID as an inductor in parallel with the resistance quantum $R \approx 50\Omega$, such that

$$Re[Z_t(\omega)] = Re[(\frac{1}{R} + \frac{1}{iX_L})^{-1}] = Re[\frac{iRX_L}{R + iX_L}] = \frac{RX_L^2}{R^2 + X_L^2}, \tag{26}$$

where $X_L = \omega L_t$ for total inductance of the SQUID junctions $L_t = (\frac{1}{L_1} + \frac{1}{L_2})^{-1}$.
In conclusion, we calculate the relaxation rates for each of the excited states:

[2]See p370-375 of M. H. Devoret's Quantum Fluctuations in Electrical Circuits, and Section 4 of Irreversibility and Generalized Noise.

$$\Gamma_1^{01} = \Gamma_1^{0\to1} + \Gamma_1^{1\to0} = \frac{1}{\hbar^2}|\langle 0|\hat{A}|1\rangle|^2(S_{\Phi_n}(w_{01}) + S_{\Phi_n}(-w_{01})) = (T_1^{01})^{-1} \tag{27a}$$

$$\Gamma_1^{12} = \Gamma_1^{1\to2} + \Gamma_1^{2\to1} = \frac{1}{\hbar^2}|\langle 1|\hat{A}|2\rangle|^2(S_{\Phi_n}(w_{12}) + S_{\Phi_n}(-w_{12})) = (T_1^{12})^{-1} \tag{27b}$$

We can maximize T_1 through $\hat{A}$, by applying a integer number of flux quanta to the environment. Minimizing T_1 would require a half-integer of flux quanta as $\hat{A} \to 0$ in the case of a symmetric SQUID ($d = 0$).

II) To address the entire transmon decoherence, we may model it by an LC oscillator with inductance $L \approx \frac{\hbar^2}{4e^2 E_J}$, capacitance $C \approx \frac{e^2}{2E_C}$. This model has time dependent charge $Q(t) = Q_o \cos(\omega t)$, with $\omega = 1/\sqrt{LC}$. Assuming energy stored in the circuit E $\hbar\omega$, $Q_o = \sqrt{2C\hbar\omega}$, we have current $I(t) = -I_o \sin(\omega t)$ for $I_o = \omega\sqrt{2C\hbar\omega}$.
The mutual inductance and oscillating current induces a voltage in the flux bias circuit as

$$V_{ind}(t) = V_o \sin(\omega t) \tag{28}$$

where $V_o = M'\omega^2\sqrt{2C\hbar\omega}$.
Environmental impedance on the order of 50Ω dissipates the average power as

$$P = \frac{V_o^2}{2R}. \tag{29}$$

Decoherence time is then approximated by

$$T_1 \approx \frac{\hbar\omega}{P} = \frac{R}{M'^2\omega^4 C} = \frac{RC}{\eta^2} \tag{30}$$

for $\eta = M'/L$ is the effective coupling strength in units of Josephson inductance.

Estimates for Qutrit Dephasing Time (T_2)

Dephasing of a quantum state can be easily understood as fluctuations in the systems energy levels, onset by noise in the Hamiltonain's external parameters. For a transmon, the primary contributions so such fluctuations are noise in charge, critical current, and magnetic flux.

Charge Noise

The transmon's sensitivity to charge noise can be expressed in terms of the differential charge dispersion $\partial E_{ij}/\partial n_g$, where i and j specify energy levels for which we are interested in their difference $E_j - E_i$. In a qutrit like ours, we care about $\partial E_{01}/\partial n_g$ and $\partial E_{12}/\partial n_g$ which give $\partial E_{02}/\partial n_g$ by linearity. Tranmon experiments have confirmed their charge insensitivity, implying that large fluctuations of the offset charge parameter typically occur at times well exceeding the acquisition time of a sing experiment. Simultaneously, small fluctuations contribute to dephasing so both need to be considered.

Dephasing due to parameter fluctuations may be expressed as decay of the off diagonal density matrix. Assuming a Gaussian distribution of noise, smaller fluctuations exist on a $1/f$ spectrum, so Eq (58) captures the realistic behavior of the density matrix (see Appendix B). Approximating the ln term to a constant, the decay time of the exponential is

$$T_2^{i\to j} \approx \frac{\hbar}{A_{n_g}}\left|\frac{\partial E_{ij}}{\partial n_g}\right|^{-1} \approx \frac{\hbar}{A_{n_g}\pi|\epsilon_j|} \tag{31}$$

where we have used

$$\frac{\delta E_{ij}}{\partial n_g} \approx \pi\epsilon_j \sin(2\pi n_g) \tag{32}$$

for the charge dispersion relation of the jth energy level

$$\epsilon_j \approx (-1)^j E_C \frac{2^{4j+5}}{j!}\sqrt{\frac{2}{\pi}}\left(\frac{E_J}{2E_C}\right)^{\frac{j}{2}+\frac{3}{4}} e^{-\sqrt{8E_J/E_C}}. \tag{33}$$

For slow charge fluctuations over longer time scales, we cannot treat the Hamiltonian with perturbation, rather we write the Hamiltonian with an oscillating term dependent of the charge offset. The first three energy levels are described by the two Hamiltonians

$$\hat{H} \approx \left[\Omega + \frac{\varepsilon}{2}\cos(2\pi n_g + 2\pi\delta n_g(t))\right]\cdot S_z \tag{34}$$

where

$$\Omega = \begin{pmatrix} \omega_{01} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_{12} \end{pmatrix} \quad , \quad \varepsilon = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix} \quad \text{and} \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{35}$$

for unperturbed energy spectrum of our qutrit eigenbasis $\hbar\omega_{if} \equiv \langle f|\vec{H}_0(\lambda_0)|f\rangle - \langle i|\vec{H}_0(\lambda_0)|i\rangle$. This gives the decay laws

$$\rho_{01}(t) \approx e^{i\omega_{01}t}\langle \exp\left[-i\frac{\epsilon_1}{2\hbar}\int_0^t dt' \cos[2\pi(n_g + \delta n_g(t))]\right]\rangle \tag{36a}$$

$$\rho_{12}(t) \approx e^{i\omega_{12}t}\langle \exp\left[-i\frac{\epsilon_2}{2\hbar}\int_0^t dt' \cos[2\pi(n_g + \delta n_g(t))]\right]\rangle. \tag{36b}$$

In precaution of a worst-case scenario, let's assume the effective charge noise fluctuates between [0,1] with a uniform probability distribution. This gives

$$\rho_{ij} \approx e^{i\omega_{ij}t}\int_0^1 dn_g \exp[-i\epsilon_j t\cos(2\pi n_g)/2\hbar] = e^{i\omega_{ij}t} J_0(|\epsilon_j|t/2\hbar) \tag{37}$$

where J_0 is the envelope of the Bessel function. $J_0(z)$ asymptotically falls off as $\sqrt{2/\pi z}$, so using the ratio $1/e$ to measure the dephasing time, e.g the decay of the envelope, is

$$T_2^{i\to j} \approx \frac{4\hbar}{e^2\pi|\epsilon_j|} \tag{38}$$

Flux Noise

Noise in the applied flux leads to fluctuations in the Josephson energy E_J. This noise is particularly of interest to us because flux is used to tune the coupler transmon transition frequency. For simplicity, we may estimate the eigenenergies with

$$E_m \approx -E_J + \sqrt{8E_cE_J}(m + 1/2) - \frac{E_c}{12}(6m^2 + 6m + 3) \tag{39}$$

which gives

$$E_{01} = \sqrt{8E_cE_J} - E_c \quad \text{and} \quad E_{12} = \sqrt{8E_cE_J} - 2E_c. \tag{40}$$

In general, multiple junctions can contribute to the Josephson energy, so we have to generalize it in terms of the applied flux,

$$E_J = E_{J\Sigma}\cos\left(\frac{\pi\Phi}{\Phi_0}\right)\sqrt{1 + d^2\tan\left(\frac{\pi\Phi}{\Phi_0}\right)} \tag{41}$$

In the symmetric junction case $d = 0$, for small flux noise about an integer multiple of Φ_0, the first derivative goes to 0, implies the dephasing is dominated by the second order, or

$$T_2^{i\to j} \approx \frac{\hbar}{A_\Phi}\left|\frac{\partial E_{ij}}{\partial \Phi}\right|^{-1} = \frac{\hbar\Phi_0}{A_\Phi \pi}\left(2E_C E_{J\Sigma}\left|\sin\frac{\pi\Phi}{\Phi_0}\tan\frac{\pi\Phi}{\Phi_0}\right|\right)^{-1/2} \tag{42}$$

in the transmon regime. Clearly, the device must be operated away from half-integer numbers of flux quantum or T_2 goes to 0. On the other hand, at integer number of flux quanta T_2 goes to infinity. This is called the flux sweet spot, where second-order contributions dominate. These terms are neglected in our treatment of noise in Appendix B but are significant as they describe the real behavior that we don't get with just the first order approximation in the flux case. These second terms, approximated about $\Phi = n\Phi_0$ for integer n

$$T_2^{i\to j} \approx \left|\frac{\pi^2 A_\Phi^2}{\hbar}\frac{\partial^2 E_{ij}}{\partial \Phi^2}\right|_{\Phi=0}^{-1} = \frac{\hbar\Phi_0^2}{A_\Phi^2 \pi^4 \sqrt{2E_{J\Sigma}E_C}} \tag{43}$$

When $d \neq 0$, as in all realistic SQUIDs, we have

$$T_2 \approx \frac{\hbar\Phi_0^2}{A_\Phi^2 \pi^4 \sqrt{2E_{J\Sigma}E_C}|d^2-1|}. \tag{44}$$

Critical Current Noise

Noise of the critical current is a secondary contribution to fluctuations of the Josephson energy. This decoherence source is believed to be due to generation and recombination of charges to ions trapped in the tunneling junction dielectric. From the Josephson energy-current relation, $E_J = I_c\hbar/2e$, the corresponding dephasing time for the transmon is given by

$$T_2^{i\to j} \approx \frac{\hbar}{A_{I_c}}\left|\frac{\partial E_{ij}}{\partial I_c}\right|^{-1} = \frac{\hbar}{A_{I_c}}\left|\frac{\partial}{\partial I_c}\sqrt{8E_c E_J}|_{E_J=I_c\hbar/2e}\right|^{-1} = \frac{\hbar}{A_{I_c}}\left|\sqrt{E_c\hbar/eI_c}\right|^{-1} = \frac{2\hbar}{\tilde{A}_{I_c}E_{ij}} \tag{45}$$

where $\tilde{A} = A_{I_c}/I_c$ gives the fluctuation amplitude.

Appendix A: Relaxation due to Noise Perturbation

We are interested with the relaxation time between the ground, first and second excited states. Extending the approach of Schoelkopf et al (SPIE 2003) to a 3^n Hilbert space, we may use first-order time-dependent perturbation theory to relate the relaxation rate to the power spectrum. First, we map the three energy levels of the qutrit to those of a fictitious spin-1 particle, i.e. the Cooper-pair box is restricted to the manifold $\{|0\rangle, |1\rangle, |2\rangle\}$. Spin +1 represents the ground state $|g\rangle$, spin 0 the first excited state $|e_1\rangle$, and spin -1 the second excited state $|e_2\rangle$. The Hamiltonian is

$$H_0 = -\hbar(\omega_{01}|g\rangle\langle g| - \omega_{12}|e_2\rangle\langle e_2|) = -\Omega S_z \tag{46}$$

With this analogy, we seek to calculate the rate of 'spin-flip' transitions induced by external noise sources. Suppose a general time dependent perturbation by an arbitrary Hermitian operator, $\hat{V}(t)$ of the form

$$\hat{V}(t) = f(t)\begin{pmatrix} 0 & V_{01} & 0 \\ V_{10} & 0 & V_{12} \\ 0 & V_{21} & 0 \end{pmatrix} = f(t)[V_{10}|e_1\rangle\langle g| + V_{01}|g\rangle\langle e_1| + V_{21}|e_2\rangle\langle e_1| + V_{12}|e_1\rangle\langle e_2|]. \tag{47}$$

Notice that this perturbation does not allow for direct transitions between $|e_2\rangle$ and $|g\rangle$, rather these require so-called 2-photon interactions. For small enough coupling terms, we may assume perturbations of first order. Let the state of the system be

$$|\Psi(t)\rangle = \begin{pmatrix} \alpha_g(t) \\ \alpha_{e_1}(t) \\ \alpha_{e_2}(t) \end{pmatrix}. \tag{48}$$

Using perturbation theory,

$$\alpha_{f,i} = -\frac{i}{\hbar}\int_0^t d\tau e^{i\omega_{f,i}\tau}\langle f|\hat{V}|i\rangle f(\tau) + O(A^2). \tag{49}$$

We are interested decay time of our qutrit, so suppose we begin in the first excited state. Then the amplitude to find the particle in the ground state is

$$\alpha_{g,e_1} = -\frac{iV_{01}}{\hbar}\int_0^t d\tau e^{i\omega_{01}\tau} f(\tau). \tag{50}$$

Similarly for the second excited state to the first excited state

$$\alpha_{e_1,e_2} = -\frac{iV_{12}}{\hbar}\int_0^t d\tau e^{i\omega_{12}\tau} f(\tau). \tag{51}$$

Because we are restricted to 2-photon interactions, the direct amplitude from the second excited state to the ground state is $\langle 0|\hat{V}(\tau)|2\rangle = 0$. To get the probability of such an interaction we can take the probability $p_{02} = p_{01}p_{12}$. We compute the probability of these transitions as

$$p_{f,i}(t) \equiv |\alpha_{f,i}|^2 = \frac{1}{\hbar^2}\int_0^t\int_0^t d\tau_1 d\tau_2 e^{-i\omega_{f,i}(\tau_1-\tau_2)}|\langle f|\hat{V}|i\rangle|^2 f(\tau_1)f(\tau_2) + O(A^3). \tag{52}$$

Although we are only concerned with the average time evolution of the system, given by

$$\overline{p}_{f,i}(t) = \frac{1}{\hbar^2}\int_0^t\int_0^t d\tau_1 d\tau_2 e^{-i\omega_{f,i}(\tau_1-\tau_2)}|\langle f|\hat{V}|i\rangle|^2\langle f(\tau_1)f(\tau_2)\rangle + O(A^3). \tag{53}$$

After a change of basis, $\tau = \tau_1 - \tau_2$ and $T = (\tau_1 + \tau_2)/2$,

$$\overline{p}_{,f}(t) = \frac{1}{\hbar^2}\int_0^t dT\int_{-B(t)}^{B(t)} d\tau e^{-i\omega_{f,i}(\tau)}|\langle f|\hat{V}|i\rangle|^2\langle f(T+\tau/2)f(T-\tau/2)\rangle + O(A^3). \tag{54}$$

where

$$B(T) = T \text{ if } T < t/2$$
$$= t - T \text{ if } T > t/2.$$

If the noise correlation function is time translation invariant and has a small, finite autocorrelation time τ_f, then for $t >> \tau_f$ the bound $B(T) \to \infty$, giving

$$\overline{p}_{f,i}(t) = \frac{1}{\hbar^2}|\langle f|\hat{V}|i\rangle|^2 \int_0^t dT \int_{-\infty}^{\infty} d\tau e^{-i\omega_{f,i}\tau} \langle f(\tau)f(0)\rangle. \tag{55}$$

We define the noise spectral density as the Fourier transform of the correlation function

$$S_f(\omega) = \int_{-\infty}^{+\infty} d\tau e^{i\omega\tau} \langle f(\tau)f(0)\rangle, \tag{56}$$

which gives the probability of transition between adjacent states:

$$\overline{p}_{f,i} = t\frac{1}{\hbar^2}|\langle f|\hat{V}|i\rangle|^2 S_f(\omega_{f,i}). \tag{57}$$

The time derivative of the probability gives the transition rate

$$\Gamma_{f,i} = \frac{1}{\hbar^2}|\langle f|\hat{V}|i\rangle|^2 S_f(\omega_{f,i}), \tag{58}$$

where we have $S_f(+\omega_{f,i})$ instead o f$-\omega_{f,i}$ because the qutrit is decaying, instead exciting. In the case that the particle begins in the ground state, and we seek the rate of transition to the excited state under a perturbation, we may reverse the frequency in the spectral density and perform an identical algebraic procedure.

An interesting, but perhaps irrelevant, property to note is that the rate of relaxation between two non-adjacent energy states is linear in time. To calculate the probability between non adjacent states, we need to multiply the probability of a two step process, i.e. $\overline{p}_{02} = \overline{p}_{01}\overline{p}_{12}$. Therefore

$$\Gamma_{02} = \frac{\partial \overline{p}_{02}}{\partial t} = \frac{\partial \overline{p}_{01}}{\partial t}\overline{p}_{12} + \frac{\partial \overline{p}_{12}}{\partial t}\overline{p}_{01} = \frac{t}{\hbar^2}|\langle 0|\hat{V}|1\rangle|^2 S_f(\omega_{0,1})|\langle 1|\hat{V}|2\rangle|^2 S_f(\omega_{1,2}) = t\Gamma_{01}\Gamma_{12} \tag{59}$$

Appendix B: Pure Dephasing due to Adiabatic Noise

Consider a general Hamiltonian for a three-level system, similar to Eq. (1). We start by expanding the Hamiltonian with a second order Taylor expansion about the perturbation $\delta\lambda$:

$$\hat{H} = -\left[\vec{H}_0(\lambda_0) + \frac{\partial \vec{H}_0}{\partial \lambda}\delta\lambda + \frac{\partial^2 \vec{H}_0}{\partial \lambda^2}\frac{\delta\lambda^2}{2} + \cdots\right]\vec{\hat{S}} \tag{60}$$

where $\vec{\hat{S}}$ contains the three spin-1 Pauli matrices (e.g. S_z is defined in Appendix A). Defining $\vec{D}_{\lambda,i} \equiv 1/\hbar\ \partial^i \vec{H}_0/\partial\lambda^i$, we obtain the eigenbasis of $\vec{H}_0(\lambda_0)\vec{\hat{S}}$,

$$\hat{H} = -(\Omega S_z + \delta\Omega_z S_z + \delta\Omega_\perp S_\perp). \tag{61}$$

The perturbed energy is given by $\delta\Omega_z \equiv D_{\lambda,1,z}\delta\lambda + D_{\lambda,2,z}\delta\lambda/2 + \cdots$, and the contribution of energy due to transverse terms (i.e. S_x and S_y) is $\delta\Omega_\perp \equiv D_{\lambda,\perp}\delta\lambda + \cdots$. The terms which dominate decoherence are related to the derivatives of $\Omega(\lambda)$ as

$$D_{\lambda,1,z} \equiv \frac{\partial\Omega}{\partial\lambda} \tag{62}$$

and

$$D_{\lambda,2,z} \equiv \frac{\partial^2\Omega}{\partial\lambda^2} - D^2_{\lambda\perp}(\Omega S)^{-1}. \tag{63}$$

The Block-Redfield theory describes the dephasing rate in terms of the relaxation rate and the 'pure' dephasing due to (short-correlated and weak) white noise as

$$\Gamma_2 = \frac{1}{2}\Gamma_1 + \Gamma_\varphi. \tag{64}$$

For noise which is **linearly coupled** to the qubit, that is $\partial\Omega/\partial\lambda \neq 0$, Bloch-Redfield theory describes the 'pure' dephasing as

$$\Gamma_\varphi = \pi S_{\delta\Omega_z}(\omega = 0) = \pi D^2_{\lambda,1,z} S_\lambda(\omega = 0) \tag{65}$$

which resembles the Golden-rule type equations from Appendix A when the 'pure' dephasing consists of noise near $\omega \approx 0$.

For noise spectral density isolated to **low frequencies**, a more intricate approach may be taken. Assuming Gaussian noise, the random phase accumulated at time t

$$\Delta\phi = D_{\lambda,1,z}\int_0^t dt'\delta\lambda(t'). \tag{66}$$

The resulting dephasing behavior is analogous to that of the famous Ramsey experiment, in which 2 $\pi/2$ pulses interact with a qubit and the experimental parameter is the delay between the pulses. The pulses are given a detuning from the qubit's resonance frequency such that the measured signal oscillates over the delay period. Further, the measured signal will also decay over time with a characteristic timescale, which gives us T_2. The decay law of the Ramsey signal is given by

$$f_z(t) = \langle e^{i\Delta\phi}\rangle = e^{-\langle\Delta\phi^2\rangle/2} = \exp\left[-\frac{t^2}{2}D^2_{\lambda,1,z}\int_{-\infty}^{+\infty} d\omega S_\lambda(\omega) sinc^2\frac{\omega t}{2}\right] \tag{67}$$

$$= \exp\left[-\frac{1}{2}D^2_{\lambda,1,z}\int_{-\infty}^{+\infty} d\omega S_\lambda(\omega)\frac{sin^2(\omega t/2)}{(\omega/2)^2}\right].$$

If the noise has a **1/f spectrum**, and we assume the $1/f$ law extends in a wide range of frequencies bounded by a lower (infrared) cut-off ω_{ir}, and an upper (ultraviolet) cut-off ω_{uv}, then

$$S_\lambda = \frac{A}{|\omega|}, \qquad \omega_{ir} < |\omega| < \omega_{uv} \tag{68}$$

where A parameterized the amplitude of energy fluctuations. At times $t << 1/\omega_{ir}$ the Ramsey decay is dominated by frequencies $\omega < 1/t$ which can be approximated as a quasistatic contribution characterized by

$$f_z(t) = \exp\left[-t^2 D^2_{\lambda,1,z} A \left(\ln\frac{1}{\omega_{ir} t} + C\right)\right]. \tag{69}$$

Generally, applied adiabatic noise on the external parameters of our Hamiltonian, or those which cause dephasing, can be expressed in rate of decay of the off-diagonal density matrix elements. These elements are, for $i \neq j$,

$$\rho_{ij}(t) = e^{i\omega_{ij}t}\langle f_z(t)\rangle = e^{i\omega_{ij}t}\langle e^{-i\int_0^t dt' v(t')}\rangle \tag{70}$$

where $v(t) = \sum_k D_{\lambda_k}\delta\lambda_k = \sum_k \frac{\partial h_z(\{\lambda_i\})}{\hbar\partial\lambda_k}\delta\lambda_k$ captures the propagation of errors from all external variables onto our computational basis. Once again assuming Gaussian noise, the power spectral density is

$$S_v(\omega) = \int_{-\infty}^{+\infty} d\tau \langle v(0)v(\tau)\rangle e^{i\omega\tau} = \sum_k \frac{\partial h_z(\{\lambda_i\})}{\hbar\partial\lambda_k} S_{\lambda_k}(\omega). \tag{71}$$

This yields a density matrix that is a function of Eq (50),

$$\rho_{ij}(t) = e^{i\omega_{ij}t}\exp\left[-\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}S_v(\omega)\frac{\sin^2(\omega t/2)}{(\omega/2)^2}\right]. \tag{72}$$

The effect of noise depends greatly on it's autocorrelation time t_c. For small autocorrelation times compared to the time needed for the system's manipulation and measurement, $t_c << t$, we may make the white noise approximation. That is, for noise spectra with a regular low-frequency behavior, the density matrix obeys an exponential law

$$\rho_{ij}(t) \approx e^{i\omega_{ij}t}\exp\left[-\frac{1}{2}|t| S_v(\omega = 0)\right]. \tag{73}$$

Finally, if the noise follows a $1/f$ spectrum, then the density matrix takes the form

$$\rho_{ij}(t) \approx e^{i\omega_{ij}t}\exp\left[-\frac{At^2}{\hbar^2}(\frac{\partial h(\omega}{\partial\lambda_i})^2 \ln\omega_{ir} t\right]. \tag{74}$$