Introduction

In this document we show that the Hamiltonian behavior of two transmon qubits coupled via a tunable resonator may be identically modeled using either floating transmons qubits as in McKay et al, 2016 (10.1103/PhysRevApplied.6.064007) and hereafter referred to as the 'floating' case, or grounded xmon qubits as in Yan et al, 2018 (10.1103/PhysRevApplied.10.054062), hereafter referred to as the 'grounded' case, despite these works using distinguishable circuits. Specifically, we derive the capacitance matrix of each circuit, and show that when either Hamiltonian is expressed in terms of such a matrix, the Hamiltonians are in fact the same.

Analytical Derivation of Circuit Hamiltonian

Consider two arbitrary superconducting qubits coupled via a tunable bus. The canonical position coordinate for these systems is the phase at each node, denoted φ_i, which may be summarized in the vector $\vec{\varphi}$. We will distinguish this from the flux across the Josephson junctions, which will be denoted $\phi_{J_i} = \varphi_{i+1} - \varphi_i$.

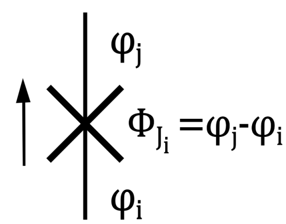

FIGURE 1. The coordinates of our system are in the convention $\phi_{J_i} = \varphi_{i+1} - \varphi_i$

We may write the Lagrangian of this three qubit system as

$$\mathscr{L} = K - U \tag{1}$$

for

$$K = \frac{1}{2}\phi_o^2 \dot{\vec{\varphi}}^\top \mathbf{C} \dot{\vec{\varphi}} \tag{2}$$

and

$$U = \sum_i E_{J_i}(1 - \cos\phi_{J_i}) \tag{3}$$

To get the Hamiltonian of this system, we take the Legendre transformation into the conjugate momentum coordinate,

$$\mathscr{H} = \vec{q}^\top \dot{\vec{\varphi}} - \mathscr{L} \tag{4}$$

for

$$\vec{q} = \frac{\partial L}{\partial \dot{\vec{\varphi}}} \tag{5}$$

By taking advantage of the symmetry of $\mathbf{C}$, it can be easily shown that

$$\frac{\partial L}{\partial \dot{\vec{\varphi}}} = \phi_o^2 \mathbf{C} \dot{\vec{\varphi}}. \tag{6}$$

This gives

$$\dot{\vec{\varphi}} = \frac{1}{\phi_o^2}\mathbf{C}^{-1}\vec{q} \tag{7}$$

$$\therefore \mathscr{H} = \frac{1}{\phi_o^2}\vec{q}^\top \mathbf{C}^{-1}\vec{q} - \frac{1}{2}\dot{\vec{\varphi}}^\top \vec{q} + U = \frac{1}{2\phi_o^2}\vec{q}^\top \mathbf{C}^{-1}\vec{q} + U. \tag{8}$$

Mapping Capacitances

Because these devices share a Hamiltonian, they exhibit identical physics in theory, so long as there is a valid mapping between them. Specifically, showing that two superconducting circuits are identical in physical behavior comes down to mapping their inverse capacitance matrices.

We will shortly note that these matrices are of different size, implying that constraints must be enacted on the larger one to reduce the number of coordinates to that of the smaller. We will perform a transformation into a unique basis that is analogous to describing the behavior of coupled harmonic oscillators, not by their individual locations, but by a superposition of harmonic modes, effectively reducing the number of parameters with which we describe the system.

We begin with the grounded case as it's easier to digest. Consider the circuit diagram from Yan et al:

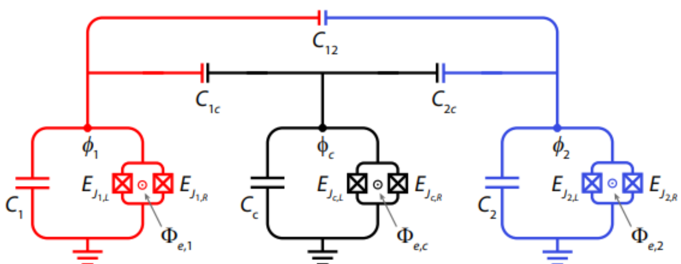

FIGURE 2. Superconducting circuit from Yan et al [2]. The circuit is implemented using two grounded 'xmon' devices split by a tunable resonator.

The qubits and coupler are grounded, so the effective capacitance only exists within each component, i.e. C_1, C_c, and C_2, between nearest neighbor components, C_{1c} and C_{2c}, and a minute capacitance reaches between the two qubits, C_{12}. The capacitance matrix for this scheme, using the Maxwell convention, is given as

$$\mathbf{C_{Xmon}} = \begin{pmatrix} C_1 + C_{1c} + C_{12} & -C_{1c} & -C_{12} \\ -C_{1c} & C_{1c} + C_c + C_{2c} & -C_{2c} \\ -C_{12} & -C_{2c} & C_2 + C_{2c} + C_{12} \end{pmatrix} \tag{9}$$

Which has inverse

$$\mathbf{C_{Xmon}}^{-1} = \frac{1}{\|\mathbf{C_{Xmon}}\|} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \tag{10}$$

for

$$A_{11} = C_2 C_{2c} + C_{1c}(C_2 + C_{2c}) + (C_2 + C_{2c})C_c + C_{12}(C_{1c} + C_{2c} + C_c) \tag{11}$$

$$A_{12} = C_{12}(C_{1c} + C_{2c}) + C_{1c}(C_2 + C_{2c}) \tag{12}$$

$$A_{13} = C_{1c}C_{2c} + C_{12}(C_{1c} + C_{2c} + C_c) \tag{13}$$

$$A_{22} = C_{1c}(C_2 + C_{2c}) + C_1(C_{12} + C_2 + C_{2c}) + C_{12}(C_{1c} + C_2 + C_{2c}) \tag{14}$$

$$A_{33} = C_{1c}(C_{2c} + C_c) + C_1(C_{1c} + C_{2c} + C_c) + C_{12}(C_{1c} + C_{2c} + C_c) \tag{15}$$

$$\|\mathbf{C_{Xmon}}\| = C_1C_cC_2 + C_1C_cC_{2c} + C_cC_2C_{12} + C_1C_2C_{1c} + C_1C_{12}C_c + C_1C_2C_{2c} + C_{1c}C_cC_2 + (C_1 + C_2 + C_c)(C_{1c}C_{2c} + C_{2c}C_{12} + C_{12}C_{1c}) \tag{16}$$

If we make the assumption that qubit-coupler capacitances are much smaller than the mode capacitances, but bigger than the qubit-qubit coupling capacitance (as would be required for adequate control of this system), that is $C_{12} << C_{ic} << C_i$, then the entries simplify as follows:

$$\mathbf{C_{Xmon}}^{-1} \approx \begin{pmatrix} \frac{1}{C_1} & \frac{C_{1c}}{C_1C_c} & \frac{C_{12}+(C_{1c}C_{2c})/C_c}{C_1C_2} \\ \frac{C_{1c}}{C_1C_c} & \frac{1}{C_c} & \frac{C_{2c}}{C_cC_2} \\ \frac{C_{12}+(C_{1c}C_{2c})/C_c}{C_1C_2} & \frac{C_{2c}}{C_cC_2} & \frac{1}{C_2} \end{pmatrix} \tag{17}$$

Writing the full Hamiltonian for the grounded case gives

$$\mathscr{H} = \frac{1}{2\phi_o^2}\vec{q}^\top \mathbf{C_{Xmon}}\vec{q} + U = \frac{1}{2\phi_o^2}\sum_i (q_i \sum_j C_{ij}^{-1} q_j) + U \tag{18}$$

Which has diagonal terms ($i = j$, $\phi_o = 1$, $h = 1$)

$$\frac{1}{2} q_i C_{ii}^{-1} q_i = \frac{q_i^2}{2C_i} = 4E_{C_i}\hat{n}_i^2 \tag{19}$$

where $\hat{n}_i = \frac{\hat{q}_i}{2e}$ and $E_{C_i} = \frac{e^2}{2C_i}$, and has off-diagonal terms ($i \neq j$)

$$q_i C_{ij}^{-1} q_j = 4e^2 \hat{n}_i C_{ij}^{-1} \hat{n}_j = 4\frac{C_{ij}}{\sqrt{C_iC_j}}\sqrt{E_{C_i}E_{C_j}}(\hat{n}_i\hat{n}_j) \tag{20}$$

Because $C_{ij}^{-1} = C_{ji}^{-1}$, there are two contributions from each off-diagonal terms, giving the final Hamiltonian

$$\mathscr{H} = 4E_{C_1}\hat{n}_1^2 + 4E_{C_2}\hat{n}_2^2 + 4E_{C_3}\hat{n}_c^2 + 8\frac{C_{1c}}{\sqrt{C_1C_c}}\sqrt{E_{C_1}E_{C_c}}(\hat{n}_1\hat{n}_c) + 8\frac{C_{2c}}{\sqrt{C_2C_c}}\sqrt{E_{C_2}E_{C_c}}(\hat{n}_2\hat{n}_c) + 8(1+\eta)\frac{C_{12}}{\sqrt{C_1C_2}}\sqrt{E_{C_1}E_{C_2}}(\hat{n}_1\hat{n}_2) + U \tag{21}$$

where $\eta = C_{1c}C_{2c}/C_{12}C_c$ arises from the fact that C_cC_{12} and $C_{1c}C_{c2}$ may be of the same order (see entry 3,3 of the inverse capacitance matrix). Note that the coefficients in front of the off-diagonal terms give the coupling strengths between each of the qubits:
For

$$\omega_\lambda = 8\sqrt{E_{J_\lambda}E_{C_\lambda}} \tag{22}$$

,

$$g_j = \frac{1}{2}\frac{C_{ij}}{\sqrt{C_iC_j}}\sqrt{E_{C_i}E_{C_j}}(\hat{n}_i\hat{n}_j) \tag{23}$$

Now, take the circuit diagram from McKay et al:

Because these qubits are not grounded, there exists a capacitance between each node and ground. For each of the six nodes, let this capacitance be C_{gi} for $i \in \{1, 2, 3, 4, 5, 6\}$. Let the capacitances between each node be C_{ij} for neighboring nodes i and j. We define the Capacitance Matrix $\mathbf{C_{Float}}$ using the Maxwell convention. For this circuit:

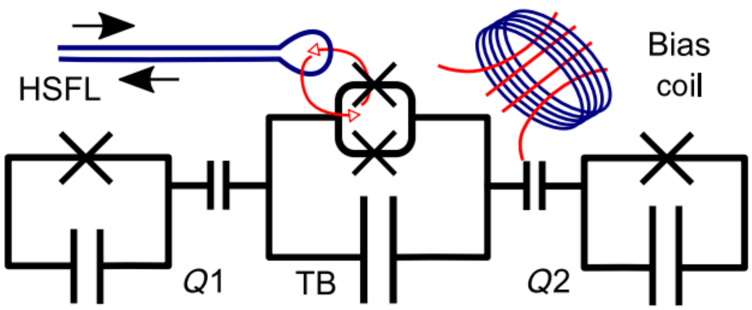

FIGURE 3. Superconducting circuit from McKay et al [1]. The circuit is implemented with two floating transmons coupled with a tunable resonator.

$$\mathbf{C_{Float}} = \begin{pmatrix} C_{g1}+C_{12} & -C_{12} & 0 & 0 & 0 & 0 \\ -C_{12} & C_{g2}+C_{12}+C_{23} & -C_{23} & 0 & 0 & 0 \\ 0 & -C_{23} & C_{g3}+C_{23}+C_{34} & -C_{34} & 0 & 0 \\ 0 & 0 & -C_{34} & C_{g4}+C_{34}+C_{45} & -C_{45} & 0 \\ 0 & 0 & 0 & -C_{45} & C_{g5}+C_{45}+C_{56} & -C_{56} \\ 0 & 0 & 0 & 0 & -C_{56} & C_{g6}+C_{56} \end{pmatrix}$$

Comparing this with the matrix $\mathbf{C_{Xmon}}$, it's obvious that size of these matrices differs, preventing us from making a direct mapping to show the circuit equivalence. To circumvent this issue, we may return to the Lagrangian, and enact a change of bases. Once again, consider the circuit Lagrangian:

$$\mathscr{L}' = \frac{\phi_o^2}{2}\dot{\vec{\varphi}}^\top \mathbf{C_{Float}} \dot{\vec{\varphi}} - U \tag{24}$$

and consider the following coordinate transformation:

$$\vec{\phi} = T\vec{\varphi} \rightarrow \vec{\varphi} = T^{-1}\vec{\phi}. \tag{25}$$

This takes our Lagrangian to

$$\mathscr{L}' = \frac{\phi_o^2}{2}\dot{\vec{\phi}}^\top (T^{-1})^\top \mathbf{C_{Float}} T^{-1}\dot{\vec{\phi}} - U = \frac{\phi_o^2}{2}\dot{\vec{\phi}}^\top \tilde{\mathbf{C}}\dot{\vec{\phi}} - U \tag{26}$$

where $\tilde{\mathbf{C}} = (T^{-1})^\top \mathbf{C_{Float}} T^{-1}$. From our previous Legendre transformation, we see that

$$\mathscr{H}' = \frac{1}{2\phi_o^2}\vec{q}^\top[(T^{-1})^\top \mathbf{C_{Float}} T^{-1}]^{-1}\vec{q} + U = \frac{1}{2\phi_o^2}\vec{q}^\top \tilde{\mathbf{C}}^{-1}\vec{q} + U \tag{27}$$

where

$$\vec{q} = \frac{\partial \mathscr{L}'}{\partial \dot{\vec{\phi}}} = \phi_o^2 (T^{-1})^\top \mathbf{C_{Float}} \dot{\vec{\varphi}} = \phi_o^2 \tilde{\mathbf{C}}\dot{\vec{\phi}}. \tag{28}$$

The variable $\vec{q}$ contains the charges on the qubit and coupler islands. What we seek to show is that through the coordinate transformation T, $\vec{q}$ contains three variables which we can map to our grounded circuit, and three offsets to our Hamiltonian, representative of the **static charges**. Because our qubits are in the transmon regime, charge dispersion is exponentially suppressed, so our energy levels and thus operating frequencies are independent of static charges which behave like an offset to our system. Therefore, static charges should not effect the behavior, much like a force step function will offset the equilibrium position of an oscillator, but the modes of the system will remain the same. To simplify the math further, let

$$\vec{Q} = \mathbf{C}\dot{\vec{\varphi}} \tag{29}$$

s.t.

$$\vec{q} = \phi_o^2 (T^{-1})^\top \vec{Q}. \tag{30}$$

Now we seek a transformation T that satisfies the following two conditions:

$$(T^{-1})^\top \vec{Q} = \begin{bmatrix} q_1 \\ * \\ q_3 \\ * \\ q_5 \\ * \end{bmatrix} \tag{31}$$

and

$$T\vec{\varphi} = \begin{bmatrix} \phi_1 \\ * \\ \phi_2 \\ * \\ \phi_3 \\ * \end{bmatrix} \tag{32}$$

where the unknown terms, denoted $*$ are arbitrarily chosen by us. This form of $\vec{\phi}$ and $\vec{q}$ express our system in terms of the charge differences across neighboring nodes, which vary with bias voltage (proportional to $\dot{\vec{\varphi}}$), whereas the static charges may be ignored as a constant offset (as mentioned above). To satisfy these conditions, it is convenient to choose (for any real numbers a, b and c)

$$(T^{-1})^\top = \begin{pmatrix} -\frac{1}{a} & \frac{a-1}{a} & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{b} & \frac{b-1}{b} & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{c} & \frac{c-1}{c} \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{33}$$

$$T^{-1} = \begin{pmatrix} -\frac{1}{a} & 1 & 0 & 0 & 0 & 0 \\ \frac{a-1}{a} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{b} & 1 & 0 & 0 \\ 0 & 0 & \frac{b-1}{b} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{c} & 1 \\ 0 & 0 & 0 & 0 & \frac{c-1}{c} & 1 \end{pmatrix} \tag{34}$$

We choose $a = b = c = 1$ for simplicity, such that

$$T = T^{-1} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{35}$$

Returning to our transformed position $\vec{\phi}$ and it's conjugate momentum $\vec{q}$, we find

$$\vec{\phi} = T\vec{\varphi} = \begin{bmatrix} \varphi_2 - \varphi_1 \\ \varphi_2 \\ \varphi_4 - \varphi_3 \\ \varphi_4 \\ \varphi_6 - \varphi_5 \\ \varphi_6 \end{bmatrix} = \begin{bmatrix} \phi_{J_1} \\ \varphi_2 \\ \phi_{J_2} \\ \varphi_4 \\ \phi_{J_3} \\ \varphi_6 \end{bmatrix} \tag{36}$$

and

$$\vec{q} = \phi_o^2 (T^{-1})^\top \vec{Q} = \phi_o^2 \begin{bmatrix} -Q_1 \\ Q_1 + Q_2 \\ -Q_3 \\ Q_3 + Q_4 \\ -Q_5 \\ Q_5 + Q_6 \end{bmatrix} \tag{37}$$

Recall the full Hamiltonian for this circuit,

$$\mathscr{H}' = \frac{1}{2\phi_o^2} \vec{q}^\top \tilde{\mathbf{C}}^{-1} \vec{q} + \sum_{i=1}^{3} -E_{J_i} \cos \phi_{J_i}. \tag{38}$$

This function has 36 charge terms of the form $q_i \tilde{\mathbf{C}}_{ij}^{-1} q_j$ for $i, j \in \{1, 2, 3, 4, 5, 6\}$, as a result of the Kinetic energy, and three flux terms as a result of the Potential. Before we analytically solve for the inverse capacitance matrix to extract the coefficients, it's would be helpful to simplify the Hamiltonian to the terms we care about. In this case, we can make a comparison to the Hamiltonian of an isolated transmon, from which we'll see an offset charge has little effect. Then, in the approximation that offset charge may be ignored, we will see that the capacitance matrix will also simplify to that of the grounded transmon.

First, consider the possible terms in the floating qubit Hamiltonian. There is the diagonal case, $i = j$, which leads to 6 Kinetic energy terms

$$K_{ii} = H_{ii} \hat{n}_i^2 \tag{39}$$

where $\hat{n}_i$ is the quantum mechanical number operator resulting from the ith charge coordinate, scaled by the single qubit charge energy H_{ii} which comes from the capacitance matrix. If $i = \{1, 3, 5\}$, then there is an associated potential term, allowing us to combine some terms of the Hamiltonian

$$\mathscr{H}_{i \in \{1,3,5\}} = H_{ii} \hat{n}_i^2 - E_{J_i} \cos \phi_{J_i} \tag{40}$$

Note: This is the Hamiltonian for an isolated transmon qubit, or of an anharmonic oscillator. It has the same form as the grounded qubits in the earlier example. Now consider the diagonal terms from $i = \{2, 4, 6\}$ in addition to the off-diagonal terms, $i \neq j$, which have Kinetic energies of the form

$$K_{ij} = H_{ij} \hat{n}_i \hat{n}_j. \tag{41}$$

which we can use to express the isolated Hamiltonians as

$$\mathscr{H}_\lambda = \tilde{H}_\lambda (\hat{n}_\lambda + \hat{n}_g)^2 - E_{J_\lambda} \cos \phi_{J_\lambda} \tag{42}$$

where $\hat{n}_g = \alpha \hat{n}_2 + \beta \hat{n}_4 + \gamma \hat{n}_6$ captures all of the terms from Eq(39) and Eq(41) when $i, j = \{2, 4, 6\}$. We map $i \in \{1, 3, 5\} \to \lambda \in \{Q1, c, Q2\}$ to indicate qubit 1, the coupler, or qubit 2, respectively. $\tilde{H}_\lambda$ is the effective charge energy resulting from the isolated transmons with a charge offset. It depends on the results of the inverse capacitance matrix, however, because we are in the transmon regime, the effect of charge noise on the eigenvalues will be exponentially suppressed. This leads to the approximation that $\tilde{H} \approx H_{ii}$.

Left to handle is the cross terms between qubits, which affect the dynamics of our system because these terms dictate the interactions between our qubits and the coupler and between the two qubits. These interations are, specifically, $\hat{n}_1 \hat{n}_3$ and $\hat{n}_5 \hat{n}_3$. In terms of our node labels, $(\lambda, c) \in \{(Q1, c), (Q2, c)\} \to (i, j) \in \{(1, 3), (5, 3)\}$

$$K_{\lambda c} = g_{\lambda c}\hat{n}_\lambda \hat{n}_c \tag{43}$$

Therefore, the floating Hamiltonian in its full form is given by

$$\begin{aligned}\mathscr{H}' &= \mathscr{H}_{Q1} + g_{1c}\hat{n}_{Q1}\hat{n}_c + \mathscr{H}_c + g_{2c}\hat{n}_{Q2}\hat{n}_c + \mathscr{H}_{Q2} \\ &= \tilde{H}_{Q1}(\hat{n}_1 + \hat{n}_g)^2 + \tilde{H}_c(\hat{n}_3 + \hat{n}_g)^2 + \tilde{H}_{Q2}(\hat{n}_5 + \hat{n}_g)^2 \\ &+ g_{1c}\hat{n}_1\hat{n}_3 + g_{2c}\hat{n}_3\hat{n}_5 - E_{J_1}\cos\phi_{J_1} - E_{J_3}\cos\phi_{J_3} - E_{J_5}\cos\phi_{J_5}.\end{aligned} \tag{44}$$

We have successfully matched this Hamiltonian with that of the grounded case. The last step is mapping the coefficients in the floating Hamiltonian, $\tilde{H}_\lambda$ and $g_{\lambda c}$ to the coefficients in the grounded Hamiltonian.

The matrix $\tilde{\mathbf{C}}^{-1}$ may be written as a 6×6 matrix with entries $\tilde{\mathbf{C}}_{ij}^{-1}$ But we only care about 9 of these entries, so effectively

$$\tilde{\mathbf{C}}^{-1} = \frac{1}{\|\tilde{\mathbf{C}}\|}\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} \end{pmatrix} \tag{45}$$

where $\|\tilde{\mathbf{C}}\|$ is the determinant of capacitance matrix. The coordinates that matter most are $\hat{n}_i\hat{n}_j$ for $i, j \in \{1, 3, 5\}$ as these determine the dynamics of the system's energy in the transmon regime. As such, we can assign $\tilde{A}_{ij} = A_{(i-1)/2,(j-1)/2}$ where A is the components of the inverse capacitance for the grounded circuit. We can make the same approximations as we did in the grounded circuit to arrive at equivalent coupling terms. These assumptions can be confirmed numerically.

Numerical Results

Consider the following circuit, representing some arbitrary configuration of three floating qubits:

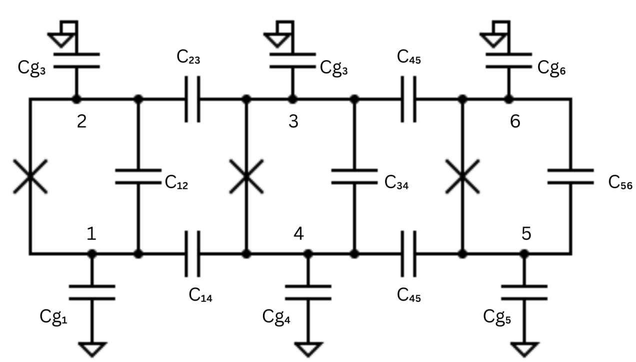

FIGURE 4. Realistically, there are capacitance terms between every neighboring node. Notice that this varies slightly from the McKay case; we have introduced C_{14} and C_{36}. Now we will show that in a certain limit, our transformed floating inverse capacitance matrix will converge to that of the grounded case.

We aimed to show that the floating and grounded Hamiltonians are equivalent by taking

$$\lim_{C_{g1},C_{g4},C_{g5}\to\infty \text{ and } C_{g2},C_{g3},C_{g6},C_{14},C_{45}\to 0} \tilde{\mathbf{C}}^{-1} = \mathbf{C}_{Yan}^{-1}$$

when assuming $C_{12}, C_{34}, C_{56} >> C_{23}, C_{36} >>$ all other capacitance omitted from the circuit, as in the grounded case.

Below are plots for the 6 different entries in our matrix.

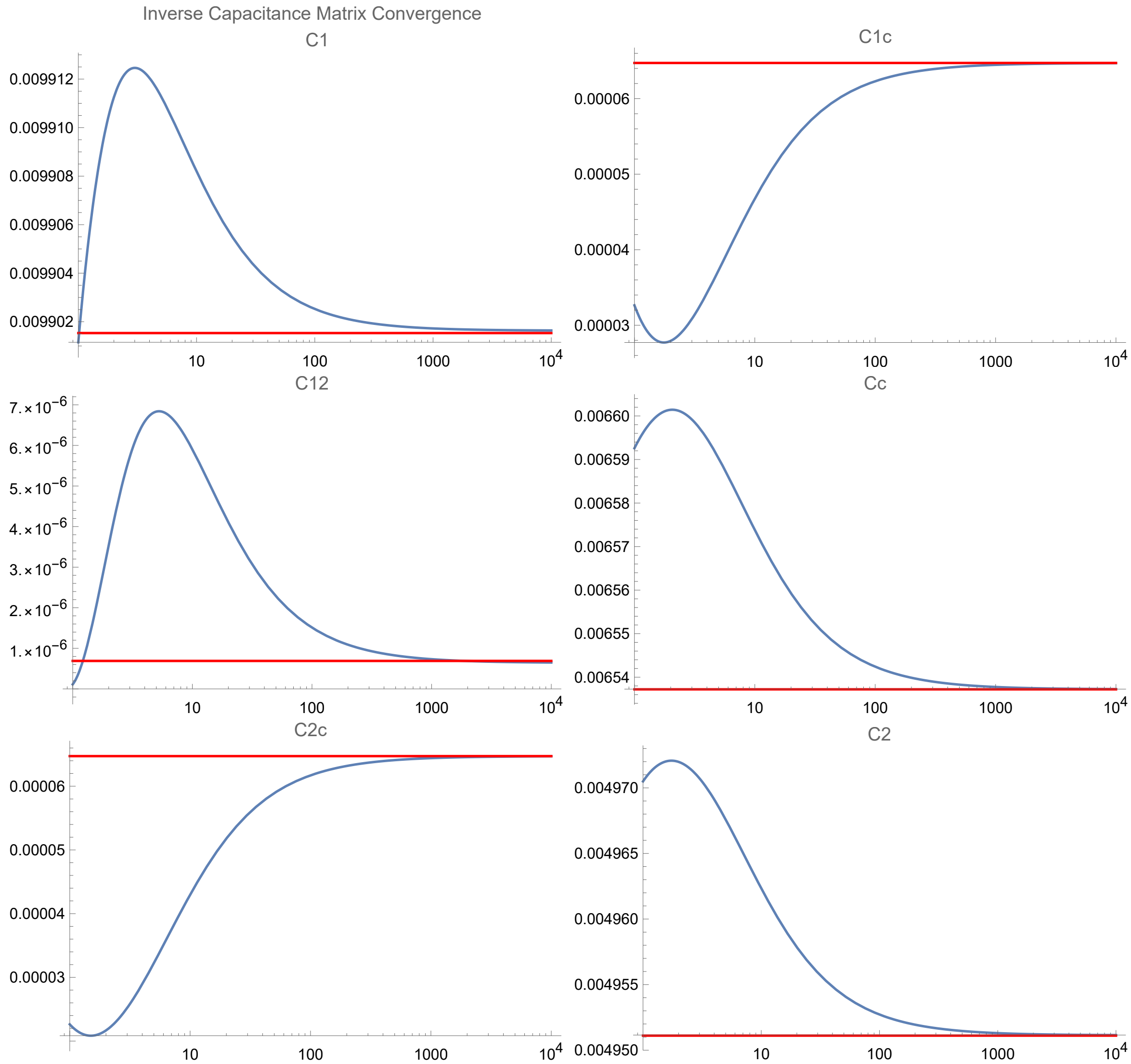

FIGURE 5. The red line is the value of the correlated entry in the grounded inverse capacitance matrix. The blue line is the inverse capacitance matrix elements from the floating circuit as a function of the limit towards infinity (x-axis). The absolute value of the coupling terms C1c and C2c are graphed. The reflection about the x-axis is a consequence of our transformation and we believe is has no physical implications.

References

[1] McKay, D. C., Filipp, S., Mezzacapo, A., Magesan, E., Chow, J. M., Gambetta, J. M. (2016). Universal Gate for Fixed-Frequency Qubits via a Tunable Bus. In Physical Review Applied (Vol. 6, Issue 6). American Physical Society (APS). https://doi.org/10.1103/physrevapplied.6.064007

[2] Yan, F., Krantz, P., Sung, Y., Kjaergaard, M., Campbell, D. L., Orlando, T. P., Gustavsson, S., Oliver, W. D. (2018). Tunable Coupling Scheme for Implementing High-Fidelity Two-Qubit Gates. In Physical Review Applied (Vol. 10, Issue 5). American Physical Society (APS). https://doi.org/10.1103/physrevapplied.10.054062